Scuola Superiore Meridionale

Mathematical and Physical Sciences for Advanced Materials and Technologies

Ph.D. Thesis

The Robin Boundary Conditions in Variational Problems and Thermal Insulation

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Contents

Preface

The content of the present PhD thesis is a *collection of selected papers* I worked on during my four-year PhD program at the Scuola Superiore Meridionale. The common theme of the problem is the *Robin boundary condition* and its application in the study of *thermal insulation*. From a mathematical point of view, all the problems we will consider are *two-phase shape optimisation problems*, where one phase, Ω, usually fixed, represents the conductor and the other one, Σ, which will vary, represents the insulator surrounding it. This thesis aims to present the various results in an organic and unified manner.

The problems can be divided mainly into two categories: *bulk* and *thin* insulation problems. For the bulk insulation problems, we mainly focus on the existence of a solution, and we will use the technique of *free discontinuity* or *free boundary* problems. In contrast, for the thin insulation problems, we will use Γ-*convergence* to *approximate the problem* in the limit of highly insulating materials of negligible thickness. The approximated problem is usually simpler and allows us to *describe the shape* of the solution through auxiliary problems.

The thesis is organised into five chapters, which we summarise below.

Chapter 1 introduces the Robin boundary condition, reviews its principal properties, and illustrates its relevance in the context of thermal insulation. Starting from the heat equation, we derive the three central mathematical problems studied in chapters 3 to 5.

- The *Capacitary problem.* We assume the steady-state temperature to be constant in the conductor Ω and harmonic in the insulator Σ. The variational energy associated with the problem is
$$E_c(\Sigma, v) = k\int_\Sigma |\nabla v|^2\,dx + \beta\int_{\partial(\bar\Omega\cup\Sigma)} v^2\,d\mathcal{H}^{n-1},$$
which is related to the relative Robin capacity of Ω with respect to $\bar\Omega\cup\Sigma$.

- The *Poisson problem.* We assume Ω to be heated by a heat source represented by a positive function f. Then the steady-state temperature will, indeed, be the solution to a *Poisson problem* with a *transmission condition* on the boundary $\partial\Omega$. In particular, we will study the variational energy
$$E_p(\Sigma, v) = \int_\Omega |\nabla v|^2\,dx + k\int_\Sigma |\nabla v|^2\,dx + \beta\int_{\partial(\bar\Omega\cup\Sigma)} v^2\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx$$

- The *spectral problem.* We look at the *eigenvalues* which govern the time evolution of the system. Such eigenvalues can be characterised by a min-max formula and the Rayleigh quotient
$$R(\Sigma, v) = \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx + k\int_\Sigma |\nabla v|^2\,dx + \beta\int_{\partial(\bar\Omega\cup\Sigma)} v^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\bar\Omega\cup\Sigma} v^2\,dx}.$$

Chapter 2 gathers the preliminary tools used throughout the thesis: properties of SBV functions, the distance function from smooth sets, tangential differentiability, and main results about Γ-convergence.

Chapter 3 is devoted to the analysis of the energy E_c. Section 3.1 presents the results of [1] (see also [28]), in which we studied a non-linear version of E_c in the setting of the SBV function. In Section 3.2, we study the energy E_c in the thin layer setting, namely, we assume $k = \varepsilon > 0$ to be small and Σ to be a thin layer, of thickness ε described by a function h. In particular, we present the results of [5], in which we studied a first-order development in Γ-convergence of the energy E_c for ε going to zero.

Chapter 4 addresses the energy E_p. Section 4.1 presents the results of [4], in which we study energy E_p in the setting of $\mathrm{SBV}^{\frac{1}{2}}$ function. In Section 4.2, we present the result of [36], in which the authors studied the limit of the energy in the thin layer setting. Then in Section 4.3, we present the result of [3], in which we proved a first-order development by Γ-convergence of the energy in the thin layer setting.

Chapter 5 studies spectral problems in the thin layer setting. In Section 5.1, we present some results about the convergence and first-order approximation of the eigenvalues (see [30]). Section 5.2 concludes the thesis with the discussion of the symmetry-breaking phenomenon in the optimisation of the principal limit eigenvalue proved in [18] (see also [38]).

Chapter 1

The Robin Boundary Condition and Thermal Insulation

In the present chapter, we will briefly introduce the *Robin boundary condition* and the optimisation problem that will be studied in the subsequent chapters.

1.1 The Robin boundary condition

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and consider the following boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_\nu u + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a prescribed function, ν is the outer unit normal to $\partial\Omega$, and $\beta \in \mathbb{R}\setminus\{\,0\,\}$ is the so called *boundary parameter*. The boundary condition in (1.1) is known as the Robin boundary condition and can be seen as a combination of *Neumann* and *Dirichlet* boundary conditions, which can be interpreted as the limit cases $\beta = 0$ and $\beta = +\infty$ respectively. Whenever we consider a solution to (1.1), unless otherwise specified, we will mean *weak solution* in the Sobolev space $H^1(\Omega)$, that is

$$\int_\Omega \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \beta u \varphi \, d\mathcal{H}^{n-1} = \int_\Omega f \varphi \, dx, \tag{1.2}$$

for all $\varphi \in H^1(\Omega)$, where $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. We notice that equation (1.2) is the *Euler-Lagrange equation* associated with the variational energy

$$F(v) = \int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} \beta v^2 \, d\mathcal{H}^{n-1} - 2\int_\Omega f v \, dx.$$

In the following, we will focus on the case $\beta > 0$ as a full discussion of the case $\beta < 0$ would require the introduction of the *Steklov eigenvalue problem.* We refer the reader to [51, §3.6] for a general discussion on the solvability of Robin boundary value problems in H^1.

The crucial tool in the study of the Robin boundary condition, in analogy with the Dirichlet case, is the following Poincaré-type inequality: there exists $C_p = C_p(\Omega, \beta) > 0$ such that for every $v \in H^1(\Omega)$

$$\int_\Omega v^2 \, dx \le C_p \left[\int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} \beta v^2 \, d\mathcal{H}^{n-1} \right]. \tag{1.3}$$

Consider the bilinear form on $H^1(\Omega)$ defined as

$$a_\beta(u, v) = \int_\Omega \nabla u \nabla \varphi \, dx + \int_{\partial\Omega} \beta u \varphi \, d\mathcal{H}^{n-1}.$$

By the continuity of the trace operator in $H^1(\Omega)$ and by the Poincaré-type inequality, it is easy to see that a_β is continuous and coercive. Moreover, if $f \in L^{\frac{2n}{n+2}}(\Omega)$ (for $n > 2$ and in $L^p(\Omega)$ with $p > 1$ for $n = 2$) the linear functional

$$\varphi \in H^1(\Omega) \mapsto \int_\Omega f\varphi\, dx \in \mathbb{R}$$

is continuous, hence, by Lax-Milgram theorem, (1.2) admits a unique solution that we will, usually, denote by u_f. We notice that, if $f \in L^2(\Omega)$, by (1.3) we can bound the $H^1(\Omega)$ norm of u_f by the $L^2(\Omega)$ norm of f. Indeed, by (1.2) and Young's inequality we have

$$\int_\Omega |\nabla u_f|^2\, dx + \int_{\partial\Omega} \beta u_f^2\, d\mathcal{H}^{n-1} = \int_\Omega f u_f\, dx \leq \frac{C_p}{2}\int_\Omega f^2\, dx + \frac{1}{2C_p}\int_\Omega u_f^2\, dx.$$

Then, using (1.3), we have

$$\int_\Omega |\nabla u_f|^2\, dx + \int_{\partial\Omega} \beta u_f^2\, d\mathcal{H}^{n-1} \leq C_p \int_\Omega f^2\, dx, \tag{1.4}$$

hence,

$$\int_\Omega u_f^2\, dx + \int_\Omega |\nabla u_f|^2\, dx \leq C_p(1 + C_p)\int_\Omega f^2\, dx. \tag{1.5}$$

From the previous inequality, we have that the resolvent operator

$$f \in L^2(\Omega) \mapsto u_f \in L^2(\Omega)$$

is compact and, by direct computations, self-adjoint and non-negative. Hence, the Laplace operator with the Robin boundary condition admits a discrete spectrum

$$0 \leq \lambda_{\beta,1}(\Omega) \leq \lambda_{\beta,2}(\Omega) \leq \cdots \leq \lambda_{\beta,j}(\Omega) \leq \cdots \to +\infty$$

and an orthonormal base of eigenfunctions $\{\, u_{\beta,j}(\Omega) \,\}$ such that

$$\begin{cases} -\Delta u_{\beta,j} = \lambda_{\beta,j}(\Omega) u_{\beta,j} & \text{in } \Omega, \\ \partial_\nu u_{\beta,j} + \beta u_{\beta,j} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

The principal Robin eigenvalue $\lambda_\beta(\Omega) = \lambda_{\beta,1}(\Omega)$ can be characterised as the minimum of the Rayleigh quotient

$$\lambda_\beta(\Omega) = \min_{v \in H^1(\Omega)\setminus\{\,0\,\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\, dx + \int_{\partial\Omega} \beta v^2\, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2\, dx}.$$

From the previous characterisation, we remark that $\lambda_\beta(\Omega)^{-1}$ is the best constant in the Poincaré-type inequality (1.3). Similarly, we can characterise the j-th eigenvalue via the min-max formula

$$\lambda_{\beta,j}(\Omega) = \min_{V \in G_j} \max_{v \in V\setminus\{\,0\,\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\, dx + \int_{\partial\Omega} \beta v^2\, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2\, dx},$$

where G_j is the set of all subspaces of $H^1(\Omega)$ of dimension j.

As the Rayleigh quotient is monotonic increasing in β, it is easy to check that the j-th eigenvalue is monotonic increasing in β as well. Moreover, if we denote by $\{\, \lambda_j^N(\Omega) \,\}$ and

$\{\,\lambda_j^D(\Omega)\,\}$ the spectrum of the Laplace operator with Neumann boundary condition and Dirichlet boundary condition, respectively, we have that for every $j \in \mathbb{N}$ and every $\beta > 0$

$$\lambda_j^N(\Omega) \le \lambda_{\beta,j}(\Omega) \le \lambda_j^D(\Omega). \tag{1.7}$$

Indeed, the Neumann and Dirichlet eigenvalues can be characterised as

$$\lambda_j^N(\Omega) = \min_{V\in G_j} \max_{v\in V\setminus\{\,0\,\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx}{\displaystyle\int_\Omega v^2\,dx}, \quad \text{and} \quad \lambda_j^D(\Omega) = \min_{V\in G_j^0} \max_{v\in V\setminus\{\,0\,\}} \frac{\displaystyle\int_\Omega |\nabla v|^2\,dx}{\displaystyle\int_\Omega v^2\,dx},$$

where G_j^0 is the set of all subspaces of $H_0^1(\Omega)$ of dimension j. The Neumann eigenvalues then correspond to the Robin ones with $\beta = 0$, and the first inequality in (1.7) is a consequence of the monotonicity in β. To prove the second inequality in (1.7), it is enough to notice that $G_j^0 \subset G_j$ and that the boundary term vanishes for every $v \in H_0^1(\Omega)$.

At the beginning of the section, we stated that the Dirichlet boundary condition can be recovered from the Robin one as the limit as the boundary parameter β goes to $+\infty$. We now have the tool needed to support such a claim. Fix $f \in L^2(\Omega)$ and for every $\beta > 0$ let u_β be the weak solution to (1.2), where this time we remark the dependence on β and not on f, which is fixed. We are interested in the limit of the functions u_β as β positively diverges. For every diverging sequence β_k, by the monotonicity of $\lambda_\beta(\Omega)$ in β, we can uniformly bound the constant $C_p(\Omega,\beta_k)$ in (1.3) so that, using (1.5) and (1.4), we have

$$\|u_{\beta_k}\|_{H^1(\Omega)} \le C,$$

and

$$\beta_k \int_{\partial\Omega} u_{\beta_k}^2\, d\mathcal{H}^{n-1} \le C \tag{1.8}$$

where the constant C is uniform in k. Then, up to a subsequence u_{β_k} converges weakly in $H^1(\Omega)$ (and strongly in $L^2(\Omega)$ and $L^2(\partial\Omega)$), to a function $u_\infty \in H^1(\Omega)$. Using (1.2) we have that for every $\varphi \in H_0^1(\Omega)$

$$\int_\Omega \nabla u_{\beta_k} \nabla\varphi\,dx = \int_\Omega f\varphi,$$

so that, by weak convergence of u_{β_k}, we have

$$\int_\Omega \nabla u_\infty \nabla\varphi\,dx = \int_\Omega f\varphi, \tag{1.9}$$

for every $\varphi \in H_0^1(\Omega)$. Moreover, by (1.8) we have that the trace of u_∞ on $\partial\Omega$ is zero, so that $u_\infty \in H_0^1(\Omega)$, and by (1.9), u_∞ is the weak solution to the Poisson problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u_\infty = f & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$

Finally, as the limit does not depend on the sequence β_k, we have that u_β converges to u_∞ strongly in $L^2(\Omega)$.

The boundary parameter β does not need to be constant and, in general, can be a non-negative function on $\partial\Omega$. The most common assumptions are $\beta \in L^\infty(\partial\Omega)$ and $\beta \ge \beta_0 > 0$ $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$, where β_0 is a constant. The first assumption implies the continuity of the bilinear form a_β, while the second implies the coercivity. To prove the existence of a unique solution to (1.1) and of the discrete spectrum, however, weaker assumptions can be used; for instance, for the continuity, it is enough to assume $\beta \in L^r(\partial\Omega)$ with $r > n-1$, while, for the coercivity one can assume $\beta \geq 0$ $\mathcal{H}^{n-1}$-almost everywhere on $\partial\Omega$ and

$$\int_{\partial\Omega_i} \beta \, d\mathcal{H}^{n-1} > 0,$$

for every connected component $\Omega_i \subseteq \Omega$. Indeed, we have the following proposition.

Proposition 1.1. *Let Ω be a bounded open set with Lipschitz boundary and let $\beta\colon \partial\Omega \to [0,\infty]$ such that*

$$\int_{\partial\Omega_i} \beta \, d\mathcal{H}^{n-1} > 0,$$

for every connected component $\Omega_i \subseteq \Omega$. Then there exists a positive constant $C_p = C_p(\Omega,\beta) > 0$ such that, for every $v \in H^1(\Omega)$

$$\int_\Omega v^2 \, dx \leq C_p \left[\int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} \beta v^2 \, d\mathcal{H}^{n-1} \right].$$

Proof. By the regularity and the boundedness of Ω, we know that it has a finite number of connected components at positive distance from one another. Hence, without loss of generality, we will prove the proposition only in the case of Ω connected.

By contradiction, assume that for every $k \in \mathbb{N}$ there exists $v_k \in H^1(\Omega)$ such that

$$\int_\Omega v_k^2 \, dx > k \left[\int_\Omega |\nabla v_k|^2 \, dx + \int_{\partial\Omega} \beta v_k^2 \, d\mathcal{H}^{n-1} \right].$$

By the homogeneity of the inequality, we can assume that

$$\int_\Omega v_k^2 \, dx = 1 \tag{1.10}$$

for every $k \in \mathbb{N}$. Then $\|v_k\|_{H^1(\Omega)}$ is bounded and, up to a subsequence, v_k converges weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ and in $L^2(\partial\Omega)$ to a function $v \in H^1(\Omega)$. Moreover, we have that

$$\lim_k \int_\Omega |\nabla v_k|^2 \, dx = 0$$

hence $\nabla v = 0$ and v is constant. Up to a further subsequence, we can assume that v_k converges to v $\mathcal{H}^{n-1}$-almost everywhere in $\partial\Omega$, so that, by Fatou's lemma

$$v^2 \int_{\partial\Omega} \beta \, d\mathcal{H}^{n-1} \leq \liminf_k \int_{\partial\Omega} \beta v_k^2 \, d\mathcal{H}^{n-1} = 0.$$

However, by (1.10) we have that $v \neq 0$, so that we can conclude

$$\int_{\partial\Omega} \beta \, d\mathcal{H}^{n-1} = 0,$$

which contradicts the assumptions. □

One could even allow the function β to be infinite on a subset Γ of the boundary with positive $\mathcal{H}^{n-1}$ measure; however, in such cases, it would be more proper to speak of *mixed boundary condition* as the right space to study the problem is

$$H^1_\Gamma(\Omega) = \left\{ v \in H^1(\Omega) \colon \ u = 0 \quad \mathcal{H}^{n-1}\text{-almost everywhere on } \Gamma \right\}.$$

A crucial difference between the Robin boundary condition and the Dirichlet one is the regularity of the set Ω. Indeed, in the previous discussion, we had to assume the boundary of Ω to be Lipschitz, while, in the case of the Dirichlet boundary condition, no particular regularity is needed. To study the Robin boundary condition on a general domain, a different approach is needed. For instance, in [35] (see also [32]), the author proves the existence of a solution to (1.1) in a general open bounded domain Ω using a key inequality by Maz'ja (see, for instance, [54, inequality 5.6.20]). The idea is to consider the Maz'ja space, that is, the abstract completion, V, of the space

$$V_0 = \left\{\, v \in H^1(\Omega) \cap C(\overline{\Omega}) \cap C^\infty(\Omega) \colon \; \|v\|_V < \infty \,\right\}$$

with respect to the norm $\|\cdot\|_V$, defined as

$$\|v\|_V^2 = \|\nabla v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\partial\Omega)}.$$

Maz'ja's inequality states that there exists $C = C(\Omega) > 0$ such that, for every $v \in V_0$,

$$\|v\|_{L^{\frac{2n}{n-1}}} \leq C\|v\|_V,$$

Then, a_β as a bilinear form on V, is continuous and coercive, so that, by Lax-Milgram theorem, equation (1.1) admits a unique solution in V for every $f \in V'$, the dual space. In particular this is true for every $f \in L^{\frac{2n}{n+1}}(\Omega)$.

From a shape optimisation point of view, Maz'ja space is not the right setting to study the Robin boundary condition when the domains are allowed to have *internal cracks*, as semincontinuity issues might arise. In such cases, at least when f is non-negative, one may use a variational approach (see, for instance, [19, 23, 22]) and define the solution to (1.1) as the minimiser of

$$F(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, dx + \beta \int_{J_v} \overline{v}^2 + \underline{v}^2 \, d\mathcal{H}^{n-1} - \int_\Omega fv \, dx$$

among the non-negative $\mathrm{SBV}(\mathbb{R}^n)$ (or $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$) functions with support in Ω an jump set $J_v \subset \partial\Omega$ (see Section 2.1 for the definitions).

We conclude the section by briefly discussing the Robin version of two classical shape optimisation problems with Dirichlet boundary condition, namely: The Faber-Krahn inequality and the Saint-Venant inequality.

The eigenvalue problem

The Faber-Krahn inequality is probably the second most famous shape optimisation problem (The first being the isoperimetric inequality) and states that the ball minimises the first Dirichlet eigenvalue among sets of equal measure, that is, for every bounded open set Ω

$$\lambda^D(\Omega^*) \leq \lambda^D(\Omega)$$

where Ω^* is a ball such that $\mathcal{L}^n(\Omega^*) = \mathcal{L}^n(\Omega)$. By the scaling properties of the Dirichlet eigenvalues, the inequality can be stated as

$$\mathcal{L}^n(B)^{\frac{2}{n}} \lambda^D(B) \leq \mathcal{L}^n(\Omega)^{\frac{2}{n}} \lambda^D(\Omega)$$

for every ball B and every open set Ω. If Ω is a (connected) open set in $\mathbb{R}^2$, the first Dirichlet eigenvalue $\lambda^D(\Omega)$ is the *principal frequency* of a Ω–shaped drum. Then the Faber-Krahn inequality states that among all drums of a given area, the circular one has the *lowest* principal frequency. Conversely, among all drums of given principal frequency, the circular one is the *cheapest* to build as it requires the least amount of material to make. The Faber-Krahn inequality was first famously conjectured by Lord Rayleigh in 1877 in his book "The Theory of Sound" ([59]), stating that:

> If the area of a membrane be given, there must evidently be some form of boundary for which the pitch (of the principal tone) is the gravest possible, and this form can be no other than the circle.

The conjecture remained open till the '20s when it was proved independently by Faber ([41]) and by Krahn ([50]). The Faber-Krahn inequality for the first Robin eigenvalue states that, for every $\beta > 0$,

$$\lambda_\beta(\Omega^*) \leq \lambda_\beta(\Omega).$$

The inequality is also known in the literature as the Bossel-Daners inequality, as it was first proved by Bossel in [15] in $n = 2$ for smooth sets and then generalised by Daners in [33] for $n \geq 3$ for open bounded sets with Lipschitz boundary. Contrary to the Dirichlet eigenvalues, the Robin ones do not have good scaling properties; hence, the Bossel-Daners inequality cannot be stated as a scaling invariant inequality.

In analogy with the Dirichlet case, the first Robin eigenvalue is described as the principal frequency of an *elastically supported membrane*; however, as we will see in the following, we will interpret $\lambda_\beta(\Omega)$ as an indicator of how fast the temperature in an Ω–shaped conductor decays. Then the Bossel-Daners inequality states that one of the reasons why many furred animals, like cats, prefer to sleep in a *curled-up shape* is that ball–shaped conductors retain heat better than any other shape of equal volume.

The Bossel-Dares inequality was generalised to arbitrary open sets in [19] (see also [20, 24]), where for a general open set with finite measure, the first Robin eigenvalue is defined as

$$\lambda_\beta(\Omega) = \inf\left\{ \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla v|^2\,dx + \beta\int_{J_v} \overline{v}^2 + \underline{v}^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{R}^n} v^2\,dx} \;\middle|\; \begin{array}{c} v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \setminus \{\,0\,\}, \\ \mathcal{L}^n(\{\, v > 0 \,\} \setminus \Omega) = 0,\ \mathcal{H}^{n-1}(J_v \setminus \partial\Omega) = 0 \end{array} \right\}.$$

In particular, it can also be stated as a Poincaré-type inequality. Namely

$$\int_{\mathbb{R}^n} |\nabla v|^2\,dx + \beta\int_{J_v} \overline{v}^2 + \underline{v}^2\,d\mathcal{H}^{n-1} \geq \lambda_\beta(\{\, v > 0 \,\}^*)\int_{\mathbb{R}^n} v^2\,dx. \tag{1.11}$$

The torsion problem

The Saint-Venant inequality is a classical inequality in shape optimisation which states that the *Torsional rigidity* of a beam of cross section Ω is always smaller than the one of a circular beam of cross section having equal area. For every bounded open set Ω, let w_Ω be the solution to the Poisson problem

$$\begin{cases} -\Delta w_\Omega = 1 & \text{in } \Omega, \\ w_\Omega = 0 & \text{on } \partial\Omega. \end{cases}$$

The torsional rigidity of a beam of cross-section Ω is defined as

$$T(\Omega) = \int_\Omega w_\Omega\,dx.$$

The Saint-Venant inequality can then be stated as

$$T(\Omega) \leq T(\Omega^*),$$

for every bounded open set Ω. The inequality was first conjectured by the French mathematician de Saint-Venant in 1856 and then formally proved by Polya in [57]. Let $\beta > 0$, for a bounded open set with Lipschitz boundary, we define we define the Robin torsional rigidity as

$$T_\beta(\Omega) = \int_\Omega w_{\beta,\Omega}\,dx,$$

where

$$\begin{cases} -\Delta w_{\beta,\Omega} = 1 & \text{in } \Omega, \\ \partial_\nu w_{\beta,\Omega} + \beta w_{\beta,\Omega} = 0 & \text{on } \partial\Omega. \end{cases}$$

The Saint-Venant inequality with Robin boundary condition, that is

$$T_\beta(\Omega) \leq T_\beta(\Omega^*),$$

was proved in [23] by studying the problem in the SBV setting. However, it is also a consequence of the Talenti comparison result for the Robin boundary condition proved in [7].

As we will see in the following section, the Saint-Venant inequality with Robin boundary condition can be interpreted as the fact that, among *uniformly heated* conductors of given volume, the ball-shaped one has the highest *mean temperature*.

1.2 Thermal insulation problems

We consider a *conductor* Ω surrounded by a *layer of insulating material* Σ. Let k_Ω and k_Σ be the *thermal diffusivities* of the two materials. If we assume the conductor to be heated by a constant-in-time *heat source*, represented by a non-negative function f, then the temperature $T = (T_\Omega, T_\Sigma)$ will satisfy the heat equation

$$\begin{cases} \partial_t T_\Omega - k_\Omega \Delta T_\Omega = f & \text{in } \Omega \times (0, +\infty), \\ \partial_t T_\Sigma - k_\Sigma \Delta T_\Sigma = 0 & \text{in } \Sigma \times (0, +\infty), \\ T_\Omega = T_\Sigma & \text{on } \partial\Omega \times (0, +\infty), \\ k_\Omega \partial_{\nu_0} T_\Omega = k_\Sigma \partial_{\nu_0} T_\Sigma & \text{on } \partial\Omega \times (0, +\infty), \\ T_\Omega(\cdot, 0) = T_{\Omega,0} & \text{in } \Omega \\ T_\Sigma(\cdot, 0) = T_{\Sigma,0} & \text{in } \Sigma, \end{cases} \tag{1.12}$$

where $T_0 = (T_{\Omega,0}, T_{\Sigma,0})$ represents the initial temperature of the insulated body, and ∂_{ν_0} is the derivative in the outward normal direction to $\partial\Omega$. To solve (1.12), a suitable *boundary condition* must be imposed on the surface $\partial\Sigma \setminus \partial\Omega$ where the insulated body comes into contact with the environment. Such boundary conditions must be chosen according to the mode of heat transfer with the environment. Let T_{ext} be the outside environment temperature. Two of the most commonly used boundary conditions are:

- the *Dirichlet* boundary condition

  $$T_\Sigma = T_{\text{ext}},$$

  which models the case of *conduction*,

- the *Robin* boundary condition

  $$-k_\Sigma \partial_\nu T_\Sigma = \beta(T_\Sigma - T_{\text{ext}}),$$

  where ∂_ν is the derivative in the outward normal direction to $\partial\Sigma \setminus \partial\Omega$, and β is usually a positive representing *heat transfer coefficient*, which, according to *Newton's law of cooling*, models the *convection* case.

More generally, by *Fourier's law*, one could impose non-linear boundary conditions of the type

$$-k_\Sigma \partial_\nu T_\Sigma = \theta(T_\Sigma, T_{\text{ext}}),$$

Such boundary conditions are relevant, for example, in the case of *radiation* heat transfer where $\theta(T_\Sigma, T_{\text{ext}})$, according to *Stefan-Boltzmann law*, is proportional to $T_\Sigma^4 - T_{\text{ext}}^4$.

In our model, the external temperature T_{ext} is assumed to be constant and homogeneous. Hence, in the following, we will consider $v = T - T_{\text{ext}}$ and change the boundary conditions accordingly.

In the present work, we will focus on the Robin boundary condition, which, in terms of v, simplifies to

$$k_\Sigma \partial_\nu v_\Sigma + \beta v_\Sigma = 0.$$

Hence, denoting the insulated body by $A = \overline{\Omega} \cup \Sigma$, we can rewrite the heat equation for our system as

$$\begin{cases} \partial_t v_\Omega - k_\Omega \Delta v_\Omega = f & \text{in } \Omega \times (0, +\infty), \\ \partial_t v_\Sigma - k_\Sigma \Delta v_\Sigma = 0 & \text{in } \Sigma \times (0, +\infty), \\ v_\Omega = v_\Sigma & \text{on } \partial\Omega \times (0, +\infty), \\ k_\Omega \partial_{\nu_0} v_\Omega = k_\Sigma \partial_{\nu_0} v_\Sigma & \text{on } \partial\Omega \times (0, +\infty), \\ k_\Sigma \partial_\nu v_\Sigma + \beta v_\Sigma = 0 & \text{on } \partial A \times (0, +\infty), \\ v(\cdot, 0) = v_0 & \text{in } A. \end{cases} \tag{1.13}$$

Let u be the *steady-state* temperature, that is the solution to the following *Poisson problem*

$$\begin{cases} -k_\Omega \Delta u_\Omega = f & \text{in } \Omega, \\ \Delta u_\Sigma = 0 & \text{in } \Sigma, \\ u_\Omega = u_\Sigma & \text{on } \partial\Omega, \\ k_\Omega \partial_{\nu_0} u_\Omega = k_\Sigma \partial_{\nu_0} u_\Sigma & \text{on } \partial\Omega \\ k_\Sigma \partial_\nu u_\Sigma + \beta u_\Sigma = 0 & \text{on } \partial A. \end{cases} \tag{1.14}$$

By the linearity of equation (1.13), $v - u$ is the solution to the homogeneous heat equation with initial condition $v_0 - u$. Hence, to determine the temperature v, we have to find the solutions to the homogeneous equation that is

$$\begin{cases} \partial_t w_\Omega - k_\Omega \Delta w_\Omega = 0 & \text{in } \Omega \times (0, +\infty), \\ \partial_t w_\Sigma - k_\Sigma \Delta w_\Sigma = 0 & \text{in } \Sigma \times (0, +\infty), \\ w_\Omega = w_\Sigma & \text{on } \partial\Omega \times (0, +\infty), \\ k_\Omega \partial_{\nu_0} w_\Omega = k_\Sigma \partial_{\nu_0} w_\Sigma & \text{on } \partial\Omega \times (0, +\infty), \\ k_\Sigma \partial_\nu w_\Sigma + \beta w_\Sigma = 0 & \text{on } \partial A \times (0, +\infty), \\ w(\cdot, 0) = w_0 & \text{in } A. \end{cases} \tag{1.15}$$

We start by looking for *particular solutions* to equation (1.15) of the type $w_\lambda(t, x) = e^{-\lambda t} u_\lambda(x)$.

By direct computations, $w_\lambda \neq 0$ is a solution if and only if

$$\begin{cases} -k_\Omega \Delta u_{\lambda,\Omega} = \lambda u_{\lambda,\Omega} & \text{in } \Omega, \\ -k_\Sigma \Delta u_{\lambda,\Sigma} = \lambda u_{\lambda,\Sigma} & \text{in } \Sigma, \\ u_{\lambda,\Omega} = u_{\lambda,\Sigma} & \text{on } \partial\Omega, \\ k_\Omega \partial_{\nu_0} u_{\lambda,\Omega} = k_\Sigma \partial_{\nu_0} u_{\lambda,\Sigma} & \text{on } \partial\Omega \\ k_\Sigma \partial_\nu u_{\lambda,\Sigma} + \beta u_{\lambda,\Sigma} = 0 & \text{on } \partial A, \end{cases} \tag{1.16}$$

that is, if and only if u_λ is an *eigenfunction* of the previous problem with *eigenvalue* λ.

By classical results, problem (1.16) admits a *discrete spectrum*

$$0 < \lambda_{\beta,1} \leq \lambda_{\beta,2} \leq \cdots \leq \lambda_{\beta,j} \leq \cdots \to +\infty.$$

Let $\{ u_j \}$ be an associated L^2-orthonormal basis of eigenfunctions, then we have

$$v(t,x) = u(x) + \sum_{j=1}^{\infty} c_j e^{-\lambda_{\beta,j} t} u_j(x),$$

where

$$c_j = \int_A (v_0 - u) u_j \, dx.$$

We remark that the eigenvalues, and in particular $\lambda_{\beta,j}$, characterise the rate of decay of v to u; in particular, smaller eigenvalues mean a *slower decay.*

Then, in designing the best possible configuration of insulating material, Σ, it is natural to study optimisation problems related to the *Poisson equation* (1.14) and to the *spectral problem* (1.16). Additionally, a third, more geometric problem arises from (1.13). Namely, the ideal case in which we continuously supply the conductor Ω with enough energy to keep it at a fixed, homogenous temperature. Up to rescaling, we can assume the temperature of Ω to be equal to one, then the steady-state temperature is a solution to

$$\begin{cases} u_\Omega = 1 & \text{in } \overline{\Omega}, \\ \Delta u_\Sigma = 0 & \text{in } \Sigma, \\ u_\Sigma = u_\Omega & \text{on } \partial\Omega, \\ k_\Sigma \partial_\nu u_\Sigma + \beta u_\Sigma = 0 & \text{on } \partial A. \end{cases} \tag{1.17}$$

We notice that (1.17) can also be seen (up to appropriate rescaling) as the limit case of equation (1.14) for k_Ω going to infinity. We will refer to this last problem as the *capacitary problem*, as the variational energy associated with the solution u is exactly the Robin relative capacity of Ω with respect to A, that is

$$\min \left\{ k_\Sigma \int_\Sigma |\nabla v|^2 \, dx + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} \;\middle|\; \begin{array}{c} v \in H^1(A), \\ v(x) = 1 \text{ in } \Omega \end{array} \right\},$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional *Hausdorff measure.*

Let us now fix some assumptions and notations. For simplicity's sake, from now on we will assume $k_\Omega = 1$ and denote k_Σ simply by k. Moreover, given the continuity condition $u_\Omega = u_\Sigma$ on $\partial\Omega$, we will drop the dependency on the set and simply denote them both with u. However, since the normal derivative, in general, does have a discontinuity on $\partial\Omega$, we will denote the limits from inside of Ω and the one from inside Σ with $\partial_\nu u^-$ and $\partial_\nu u^+$ respectively. Finally, from a shape optimisation point of view, it will be useful to explicitly express the dependence on the configuration of insulating material, which for us will usually be the *whole* insulated body, hence, for instance, in the following we will denote the steady state temperature in an insulated body A, by the symbol u_A.

In the following sections, we will discuss each problem and its respective mathematical formulation.

1.2.1 The capacitary problem

With the name *the Capacitary Problem* we refer to the following:

> given a conductor Ω kept at fixed temperature $u = 1$, find the best possible configuration of insulating material, Σ, surrounding it.

To give a mathematical formulation of the problem, we first need to specify what makes a configuration of insulating material "better" than another. As mentioned in the previous section, given a set A containing Ω, the function u_A, solution to the boundary value problem (1.17), is the minimiser of the variational energy

$$E_c(A, v) = k \int_{A \setminus \overline{\Omega}} |\nabla v|^2 \, dx + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1}$$

among all functions $v \in H^1(A)$ such that $v = 1$, almost everywhere in Ω. At its optimum, the energy is given by

$$E_c(A) = E_c(A, u_A) = \beta \int_{\partial A} u_A \, d\mathcal{H}^{n-1},$$

by the Robin boundary condition, this is equivalent to

$$E_c(A) = -k \int_{\partial A} \partial_\nu u_A \, d\mathcal{H}^{n-1},$$

That is proportional to the *heat flux* across the boundary of the insulated body. $E_c(A)$ is then related to the amount of energy needed to maintain the body at the desired temperature; hence, it is a measure of the "goodness" of the configuration. We could then say that a configuration A_1 is better than a configuration A_2 if *less energy is needed*, that is, if

$$E_c(A_1) < E_c(A_2).$$

Finally, the problem is to minimise the quantity $E_c(\cdot)$ in the class of sets containing Ω. However, the cost of the insulating material should also be taken into account, either by imposing a maximum quantity, m, at one's disposal, that is, adding the constraint

$$\mathcal{L}^n(A \setminus \Omega) \leq m,$$

or by penalising the per-unit cost of the insulating material, that is, to minimise the energy

$$\mathcal{E}_c(A) = E_c(A) + C_0 \mathcal{L}^n(A \setminus \Omega).$$

We remark that, if a minimiser, A^*, of $\mathcal{E}_c$ exists, then it is also a minimiser of the constrained formulation with mass $m = \mathcal{L}^n(A^* \setminus \Omega)$.

In Chapter 3 we will study the *existence* and the *shape* of an optimal configuration.

To prove the existence of an optimal configuration, we will use the penalised formulation as it adds compactness to the problem. Actually, following the approach of [28] (see also [22]) we will work in a weaker setting, namely, we will work in the setting of special functions of bounded variation (see Definition 2.10), identifying the set A with the set $\{ u_A > 0 \}$ (see Section 3.1 for the precise formulation).

To try to describe the shape of an optimal configuration, on the other hand, we will use the constrained formulation. In particular, we will work in the *thin layer* setting: we will assume that the insulating layer $\Sigma = A \setminus \overline{\Omega}$ can be described by a positive function h on the boundary of Ω, times a small parameter $\varepsilon > 0$, related to the *thickness* of the layer. We will be interested in the *asymptotic behaviour* of the energy as ε goes to zero (see Section 3.2 for the precise formulation).

The results by [16] and [6] for the *Dirichlet* boundary condition suggest that, in order to still see, in the limit, the effects of the insulating layer but not trivialise the problem, we should assume that the thermal diffusivity, k, is also of order ε, that is: we are using a "small" quantity of "good" insulating material.

We will approximate the energy using the tools of Γ-convergence (see Section 2.4) and optimise the approximated energies with respect to the function h.

1.2.2 The Poisson problem

With the name *the Poisson Problem* we refer to the following:

> for a given heat source, $f \geq 0$, distributed in a conductor Ω, find the best possible configuration of insulating material, Σ, surrounding it.

As for the previous section, the "best" configuration is one that minimises an appropriate energy associated with the system. The energy chosen for the capacitary problem in this case is not a good choice. Indeed, as previously stated, the heat flux across the boundary of the insulated body is related to the energy supplied to the system, that is, the integral of the function f. Mathematically, given a set A containing Ω, if u_A if the solution to the boundary value problem (1.14), then by direct computations, we have

$$-k \int_{\partial A} \partial_\nu u_A \, d\mathcal{H}^{n-1} = \int_\Omega f \, dx,$$

which is constant in A.

From a variational point of view, u_A is the minimiser, in $H^1(A)$, of the energy

$$E_p(A, v) = \int_\Omega |\nabla v|^2 \, dx + k \int_{A \setminus \overline{\Omega}} |\nabla v|^2 \, dx + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} - 2 \int_\Omega f v \, dx,$$

whose minimum is

$$E_p(A) = E_p(A, u_A) = - \int_\Omega f u_A \, dx.$$

If the heat source is uniformly distributed, that is, f is constant, then the energy $E_p(A)$ is proportional to the opposite the *average temperature* in the conductor. Then

$$E_p(A_1) < E_p(A_2)$$

means that, if we supply the two systems with the *same amount of heating*, the average temperature in the configuration A_1 will be larger than the one in A_2. Hence, the energy E_p is a good choice at least in the case where f is constant.

For a general function f, the energy E_p is the opposite of a *weighted average temperature*, where the weight is the source f itself, which is still a reasonable choice of energy to minimise and is indeed the one we will focus on.

Other suitable choices could be, for instance, the opposite of the average temperature or the L^p distance from a desired temperature. However, our variational approach in proving the existence of an optimal configuration does not apply to such energies.

In Chapter 4 we will study the *existence* and the *shape* of an optimal configuration.

To prove the existence of an optimal configuration, we will study the penalised energy

$$\mathcal{E}_p(A) = E_p(A) + C_0\mathcal{L}^n(A \setminus \Omega)$$

with an approach similar to the one mentioned in the capacitary problem.

To describe the shape of an optimal configuration, we will work in the thin layer setting and use the constrained formulation. In particular, we will illustrate the results by [36] (see also [27, 18]) about the optimisation of the limit energy, and then we will prove a first-order development by Γ-convergence of the energy E_p with respect to the parameter ε.

1.2.3 The spectral problem

With the name the *spectral problem*, we refer to the following

> given a conductor Ω find the configuration of insulating material, Σ, surrounding it, which maintains the temperature the longest.

Let A be a set containing Ω and denote by $\{\, \lambda_{\beta,j}(A) \,\}$ the spectrum of the eigenvalue problem (1.16) and let $\{\, u_A, j \,\}$ be an orthonormal basis of corresponding eigenfunctions. If v_0 is the initial temperature of the system and u_A is the steady-state temperature, as already stated, the temperature v_A can be expressed as

$$v_A(t,x) = u(x) + \sum_{j=1}^{\infty} c_j e^{-\lambda_{\beta,j}(A)t} u_{A,j}(x),$$

where the coefficients c_j are such as

$$v_0(x) - u_A(x) = \sum_{j=1}^{\infty} c_j u_{A,j}(x).$$

Then the square of the L^2-distance between $v_A(t,\cdot)$ and the steady state temperature u_A, can be written as

$$\int_A (v_A(t,x) - u_A(x))^2\,dx = \sum_{j=1}^{\infty} c_j^2 e^{-2\lambda_{\beta,j}(A)t},$$

from which we have that

$$\frac{\partial_t \|v_A(t,\cdot) - u_A\|_{2,A}}{\|v_A(t,\cdot) - u_A\|_{2,A}} \leq -\lambda_{\beta,1}(A).$$

Then $\lambda_{\beta,1}$ can be seen as a good indicator of how fast the temperature of a given configuration converges to its steady-state, hence minimising the first eigenvalue is a suitable criterion to find the configuration which maintains the temperature the longest.

Variationally, the following characterisation holds: Let

$$R_\beta(A,u) = \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx + k\int_{A\setminus\overline{\Omega}} |\nabla u|^2\,dx + \beta\int_{\partial A} u^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_A u^2\,dx},$$

then, the first eigenvalue is

$$\lambda_{\beta,1}(A) = \min_{u \in H^1(A) \setminus \{0\}} R_\beta(A, u).$$

The j-th eigenvalue can be characterised as

$$\lambda_{\beta,j}(A) = \min_{V \in G_j} \max_{u \in V \setminus \{0\}} R_\beta(A, u),$$

where we recall that G_j is the set of all subspaces of $H^1(A)$ of dimension j.

In Chapter 5, we will work in the thin layer setting to prove convergence and first-order asymptotic estimates of the eigenvalues. Moreover, we will discuss the optimisation of the limit j-th eigenvalue in the constrained formulation. Finally, we will present a surprising *symmetry breaking result* by [18] and [38] (see also [45]) which states that when Ω is a ball, the configuration of insulating material which maintains the temperature the longest is not always the radially symmetric one.

Chapter 2

Preliminaries

In the present chapter, we will introduce notions and state results that will be used throughout the thesis. We recommend that for the first reading, one omit this chapter and refer to it only when needed in the next chapters.

In Section 2.1 we define *functions of bounded variation* with particular attention to the concept of *approximate upper* and *lower limits*, and state compactness theorems in the spaces SBV and $\mathrm{SBV}^{\frac{1}{2}}$.

In Section 2.2 we state the main properties of the distance function from a set and define the *principal curvatures* of its level sets.

In Section 2.3 we give useful results to compute integrals of *hypersurfaces*.

Finally, in Section 2.4 we recall basic properties of Γ-*convergence* and of the *asymptotic development* by Γ-convergence.

2.1 Functions of bounded variation

In this section, we recall some definitions and properties of the spaces BV, SBV, and $\mathrm{SBV}^{\frac{1}{2}}$. We refer to [8], [19], [40] for a deep study of the properties of these functions.

In the following, given an open set $\Omega \subseteq \mathbb{R}^n$ and $1 \leq p \leq \infty$, we will denote the $L^p(\Omega)$ norm of a function $v \in L^p(\Omega)$ as $\|v\|_{p,\Omega}$, in particular when $\Omega = \mathbb{R}^n$ we will simply write $\|v\|_p = \|v\|_{p,\mathbb{R}^n}$.

Definition 2.1 (BV)**.** Let $u \in L^1(\mathbb{R}^n)$. We say that u is a function of *bounded variation* in $\mathbb{R}^n$ and we write $u \in \mathrm{BV}(\mathbb{R}^n)$ if its distributional derivative is a Radon measure, namely

$$\int_\Omega u\,\frac{\partial\varphi}{\partial x_i}\,dx = \int_\Omega \varphi\,dD_i u \qquad \forall\varphi \in C_c^\infty(\mathbb{R}^n),$$

with Du a $\mathbb{R}^n$-valued measure in $\mathbb{R}^n$. We denote with $|Du|$ the total variation of the measure Du. The space $\mathrm{BV}(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|u\|_{\mathrm{BV}(\mathbb{R}^n)} = \|u\|_1 + |Du|(\mathbb{R}^n).$$

Definition 2.2. Let $E \subseteq \mathbb{R}^n$ be a measurable set. We define the *set of points of density 1 for E* as

$$E^{(1)} = \left\{ x \in \mathbb{R}^n \,\middle|\, \lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} = 1 \right\},$$

and the *set of points of density 0 for E* as

$$E^{(0)} = \left\{ x \in \mathbb{R}^n \,\middle|\, \lim_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\mathcal{L}^n(B_r(x))} = 0 \right\}.$$

Moreover, we define the *essential boundary* of E as

$$\partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

Definition 2.3 (Approximate upper and lower limits)**.** Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We define the *approximate upper and lower limits* of u, respectively, as

$$\overline{u}(x) = \inf\left\{ t \in \mathbb{R} \;\middle|\; \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u > t \})}{\mathcal{L}^n(B_r(x))} = 0 \right\},$$

and

$$\underline{u}(x) = \sup\left\{ t \in \mathbb{R} \;\middle|\; \limsup_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ u < t \})}{\mathcal{L}^n(B_r(x))} = 0 \right\}.$$

We define the *jump set* of u as

$$J_u = \{ x \in \mathbb{R}^n \mid \underline{u}(x) < \overline{u}(x) \}.$$

We denote by K_u the closure of J_u.

If $\overline{u}(x) = \underline{u}(x) = l$, we say that l is the approximate limit of u as y tends to x, and we have that, for any $\varepsilon > 0$,

$$\limsup_{r\to 0^+} \frac{\mathcal{L}^n(B_r(x) \cap \{ |u - l| \geq \varepsilon) \}}{\mathcal{L}^n(B_r(x))} = 0.$$

If $u \in \mathrm{BV}(\mathbb{R}^n)$, the jump set J_u is a $(n-1)$-rectifiable set, i.e. $J_u \subseteq \bigcup_{i\in\mathbb{N}} M_i$, up to a $\mathcal{H}^{n-1}$-negligible set, with M_i a C^1-hypersurface in $\mathbb{R}^n$ for every i. We can then define $\mathcal{H}^{n-1}$-almost everywhere on J_u a normal ν_u coinciding with the normal to the hypersurfaces M_i. Furthermore, the direction of $\nu_u(x)$ is chosen in such a way that the approximate upper and lower limits of u coincide with the approximate limit of u on the half-planes

$$H^+_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \geq 0 \}$$

and

$$H^-_{\nu_u} = \{ y \in \mathbb{R}^n \mid \nu_u(x) \cdot (y - x) \leq 0 \}$$

respectively.

Definition 2.4. Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define the *relative perimeter* of E inside Ω as

$$P(E;\Omega) = \sup\left\{ \int_E \operatorname{div}\varphi\, dx \;\middle|\; \begin{array}{c} \varphi \in C^1_c(\Omega, \mathbb{R}^n) \\ |\varphi| \leq 1 \end{array} \right\}.$$

If $P(E;\mathbb{R}^n) < +\infty$ we say that E is a *set of finite perimeter.*

Theorem 2.5 (Relative Isoperimetric Inequality)**.** *Let Ω be an open, bounded, connected set with Lipschitz boundary. Then there exists a positive constants $C = C(\Omega)$ such that*

$$\min\{ \mathcal{L}^n(\Omega \cap E), \mathcal{L}^n(\Omega \setminus E) \}^{\frac{n-1}{n}} \leq C P(E;\Omega),$$

for every set E of finite perimeter.

See, for instance, [54] for the proof of this theorem.

Theorem 2.6. *Let Ω be an open, bounded, connected set with Lipschitz boundary. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) \leq C\mathcal{H}^{n-1}(\partial^* E \cap \Omega)$$

for every set of finite perimeter $E \subset \Omega$ with $0 < \mathcal{L}^n(E) \leq \mathcal{L}^n(\Omega)/2$.

We refer to [29, Theorem 2.3] for the proof of the previous theorem, observing that if Ω is a Lipschitz set, then it is an admissible set in the sense defined in [29](see [63, Remark 5.10.2]).

Theorem 2.7 (Decomposition of BV functions)**.** *Let $u \in \mathrm{BV}(\mathbb{R}^n)$. Then we have*

$$dDu = \nabla u\, dx + |\overline{u} - \underline{u}|\nu_u\, d\mathcal{H}^{n-1}\lfloor_{J_u} + dD^c u,$$

where ∇u is the density of Du with respect to the Lebesgue measure, ν_u is the normal to the jump set J_u and $D^c u$ is the Cantor part *of the measure Du. The measure $D^c u$ is singular with respect to the Lebesgue measure and concentrated outside of J_u.*

Definition 2.8. Let $v \in \mathrm{BV}(\mathbb{R}^n)$, let $\Gamma \subseteq \mathbb{R}^n$ be a $\mathcal{H}^{n-1}$-rectifiable set and let $\nu(x)$ be the generalized normal to Γ defined for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. For $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ we define the traces $\gamma_\Gamma^\pm(v)(x)$ of v on Γ by the following Lebesgue-type limit quotient relation

$$\lim_{r \to 0^+} \frac{1}{r^n} \int_{B_r^\pm(x)} |v(y) - \gamma_\Gamma^\pm(v)(x)|\, dy = 0,$$

where

$$B_r^+(x) = \{\, y \in B_r(x) \mid \nu(x) \cdot (y - x) > 0 \,\},$$
$$B_r^-(x) = \{\, y \in B_r(x) \mid \nu(x) \cdot (y - x) < 0 \,\}.$$

Remark 2.9. Notice that, by [8, Remark 3.79], for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, $(\gamma_\Gamma^+(v)(x), \gamma_\Gamma^-(v)(x))$ coincides with either $(\overline{v}(x), \underline{v}(x))$ or $(\underline{v}(x), \overline{v}(x))$, while, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus J_v$, we have that $\gamma_\Gamma^+(v)(x) = \gamma_\Gamma^-(v)(x)$ and they coincide with the approximate limit of v in x. In particular, if $\Gamma = J_v$, we have

$$\gamma_{J_v}^+(v)(x) = \overline{v}(x) \qquad \gamma_{J_v}^-(v)(x) = \underline{v}(x)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in J_v$.

We now focus our attention on the BV functions whose Cantor parts vanish.

Definition 2.10 (SBV)**.** Let $u \in \mathrm{BV}(\mathbb{R}^n)$. We say that u is a *special function of bounded variation* and we write $u \in \mathrm{SBV}(\mathbb{R}^n)$ if $D^c u = 0$.

For SBV functions, we have the following.

Theorem 2.11 (Chain rule)**.** *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then if $u \in \mathrm{SBV}(\mathbb{R}^n)$, we have*

$$\nabla g(u) = g'(u)\nabla u.$$

Furthermore, if g is increasing,

$$\overline{g(u)} = g(\overline{u}), \quad \underline{g(u)} = g(\underline{u})$$

while, if g is decreasing,

$$\overline{g(u)} = g(\underline{u}), \quad \underline{g(u)} = g(\overline{u}).$$

We now state a compactness theorem in SBV that will be useful in the following.

Theorem 2.12. *[Compactness in* SBV*] Let u_k be a sequence in* $\mathrm{SBV}(\mathbb{R}^n)$*. Let $p, q > 1$, and let $C > 0$ such that for every $k \in \mathbb{N}$*

$$\int_{\mathbb{R}^n} |\nabla u_k|^p\, dx + \|u_k\|_\infty + \mathcal{H}^{n-1}(J_{u_k}) < C.$$

Then there exists $u \in \mathrm{SBV}(\mathbb{R}^n)$ and a subsequence u_{k_j} such that

- Compactness:

$$u_{k_j} \xrightarrow{L^1_{\mathrm{loc}}(\mathbb{R}^n)} u$$

- Lower semicontinuity: *for every open set* A *we have*

$$\int_A |\nabla u|^p \, dx \leq \liminf_{j\to+\infty} \int_A |\nabla u_{k_j}|^p \, dx$$

and

$$\int_{J_u\cap A} (\overline{u}^q + \underline{u}^q) \, d\mathcal{H}^{n-1} \leq \liminf_{j\to+\infty} \int_{J_{u_{k_j}}\cap A} \left(\overline{u}^q_{k_j} + \underline{u}^q_{k_j}\right) d\mathcal{H}^{n-1}$$

We refer to [8, Theorem 4.7, Theorem 4.8, Theorem 5.22] for the proof of this theorem.
For the study of the Poisson problem, the following class of functions will be useful.

Definition 2.13 (SBV$^{1/2}$)**.** Let $u \in L^2(\mathbb{R}^n)$ be a non-negative function. We say that $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ if $u^2 \in \mathrm{SBV}(\mathbb{R}^n)$. In addition, we define

$$J_u := J_{u^2} \qquad \overline{u} := \sqrt{\overline{u^2}} \qquad \underline{u} := \sqrt{\underline{u^2}}$$

$$\nabla u := \frac{1}{2u}\nabla(u^2)\chi_{\{\,u>0\,\}}$$

Notice that this definition extends the validity of the Chain Rule to the functions in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$. We refer to [19, Lemma 3.2] for the coherence of this definition.

Theorem 2.14 (Compactness in SBV$^{1/2}$)**.** *Let* u_k *be a sequence in* $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ *and let* $C > 0$ *be such that for every* $k \in \mathbb{N}$

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \, dx + \int_{J_{u_k}} \left(\overline{u}_k^2 + \underline{u}_k^2\right) d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \, dx < C$$

Then there exists $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ *and a subsequence* u_{k_j} *such that*

- Compactness:

$$u_{k_j} \xrightarrow{L^2_{\mathrm{loc}}(\mathbb{R}^n)} u$$

- Lower semicontinuity: *for every open set* Ω *we have*

$$\int_\Omega |\nabla u|^2 \, dx \leq \liminf_{j\to+\infty} \int_\Omega |\nabla u_{k_j}|^2 \, dx$$

and

$$\int_{J_u\cap\Omega} \left(\overline{u}^2 + \underline{u}^2\right) \mathcal{H}^{n-1} \leq \liminf_{j\to+\infty} \int_{J_{u_{k_j}}\cap\Omega} \left(\overline{u}^2_{k_j} + \underline{u}^2_{k_j}\right) \mathcal{H}^{n-1}$$

We conclude this section with the following proposition, whose proof can be found in [28, Lemma 3.1].

Proposition 2.15. *Let* $u \in \mathrm{BV}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Then*

$$\int_0^1 P(\{\,u > s\,\}\,;\mathbb{R}^n \setminus J_u) \, ds = |Du|(\mathbb{R}^n \setminus J_u).$$

2.2 Distance function and curvatures

We refer to [43, Section 14.6] for the notions in this section. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary, let

$$d(x) = d(x, \Omega) = \min_{y \in \bar{\Omega}} |x - y|$$

be the distance function from Ω and define

$$\Gamma_t = \{\, x \in \mathbb{R}^n \mid d(x) < t \,\} \setminus \Omega.$$

The regularity of Ω ensures that Ω satisfies a uniform exterior ball condition, and in particular, there exists $d_0 > 0$ such that $d \in C^{1,1}(\Gamma_{d_0})$. Moreover, for every $z \in \Gamma_{d_0}$ there exists a unique $\sigma(z) \in \partial\Omega$ such that

$$|\sigma(z) - z| = d(z),$$

and we will say that $\sigma(z)$ is the *metric projection* of z over Ω; we will also denote by ν_0 the extension of the outer unit normal to $\partial\Omega$ such that for every $z \in \Gamma_{d_0}$ we have $\nu_0(z) = \nu_0(\sigma(z))$.

For every $x \in \Gamma_{d_0}$ we can write

$$x = \sigma(x) + d(x)\nu_0(x),$$

and notice that we have $\sigma, \nu_0 \in C^{0,1}(\Gamma_{d_0}; \mathbb{R}^n)$, and $\nabla d = \nu_0$. For $\mathcal{H}^{n-1}$-a.e. $\sigma \in \partial\Omega$ we can consider the Hessian matrix $D^2 d(\sigma)$, and, since $|\nabla d| = 1$, we have that $\nu_0(\sigma)$ is an eigenvector with corresponding zero eigenvalue. Let $\{\, \tau_1, \dots, \tau_{n-1}, \nu_0 \,\}$ be a system of normalized eigenvectors.

Definition 2.16. Let $\sigma \in \partial\Omega$ and let $\{\, \tau_1(\sigma), \dots, \tau_{n-1}(\sigma), \nu_0(\sigma) \,\}$ be an ordered system of normalized eigenvectors for $D^2 d(\sigma)$. We define the *principal curvatures* $k_1(\sigma), \dots, k_{n-1}(\sigma)$ of $\partial\Omega$ at the point σ as the eigenvalues of the matrix

$$D^2 d(\sigma)$$

corresponding to the eigenvectors $\tau_1(\sigma), \dots, \tau_{n-1}(\sigma)$.

Let $t \in (0, d_0)$ and consider

$$\gamma_t = \partial(\Omega \cup \Gamma_t) = \{\, x \in \mathbb{R}^n \mid d(x) = t \,\} \setminus \Omega.$$

Our regularity assumptions allow us to consider for every $x \in \Gamma_{d_0}$ the matrix $D^2 d(x)$, which is symmetric and represents the second fundamental form of $\gamma_{d(x)}$. Moreover, by direct computation (see, for instance, [43, Lemma 14.17]), we have that $\tau_i(\sigma(x))$ are eigenvectors for $D^2 d(x)$ with eigenvalues computed in the following definition.

Definition 2.17. Let $x \in \Gamma_{d_0}$. For every $i = 1, \dots, n-1$ we denote by

$$\tau_i(x) := \tau_i(\sigma(x)),$$

and we denote by $k_1(x), \dots, k_{n-1}(x)$ the corresponding sequence of eigenvalues of $D^2 d(x)$

$$k_i(x) := \frac{k_i(\sigma(x))}{1 + d(x) k_i(\sigma(x))},$$

Remark 2.18. By the properties of $D^2 d(x)$ we can diagonalize $D\sigma(x)$ as

$$D\sigma(x)\, \tau_i(\sigma) = \frac{1}{1 + d(x) k_i(\sigma)} \tau_i(\sigma).$$

2.3 Calculus on hypersurfaces

We refer to [52] for the notions in this section. For every $v, w \in \mathbb{R}^n$ let the *tensorial product* $v \otimes w$ be the unique linear operator on $\mathbb{R}^n$ such that, for every $z \in \mathbb{R}^n$,

$$(v \otimes w)z = (w \cdot z)v.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary and let ν_0 be the outer unit normal to its boundary. For every $\sigma \in \partial\Omega$ we denote by

$$T_\sigma \partial\Omega = \{\, y \in \mathbb{R}^n \mid \nu_0(\sigma) \cdot y = 0 \,\},$$

the *tangent space to* $\partial\Omega$ *at* σ. Let $\tau_1(\sigma), \dots, \tau_{n-1}(\sigma)$ be an orthonormal basis for $T_\sigma\partial\Omega$.

Definition 2.19 (Tangential gradient)**.** Let Ω be a bounded open set of class C^1, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, and let $\phi \in C^{0,1}(U;\mathbb{R}^n)$. We define for $\mathcal{H}^{n-1}$-a.e. $\sigma \in \partial\Omega$ the *tangential gradient of* ϕ at σ as the the linear map

$$D^{\partial\Omega}\phi(\sigma) \colon T_\sigma\partial\Omega \to \mathbb{R}^n$$

defined as

$$\begin{aligned} D^{\partial\Omega}\phi(\sigma) =& \sum_{h=1}^{n-1} (\nabla\phi(\sigma)\tau_h) \otimes \tau_h \\ =& \nabla\phi(\sigma) - (\nabla\phi(\sigma)\nu_0) \otimes \nu_0. \end{aligned}$$

Notice that the definition of tangential gradient does not depend on the choice of the orthonormal basis $\tau_1, \dots, \tau_{n-1}$.

Definition 2.20 (Tangential Jacobian)**.** Let Ω be a bounded open set of class C^1, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, and let $\phi \in C^{0,1}(U;\mathbb{R}^n)$. We define *the tangential Jacobian of* ϕ as

$$J^{\partial\Omega}\phi = \sqrt{\det\left((D^{\partial\Omega}\phi)^T (D^{\partial\Omega}\phi)\right)},$$

where the determinant has to be intended in the space $T_\sigma\partial\Omega \otimes T_\sigma\partial\Omega$.

Theorem 2.21 (Area formula on surfaces)**.** *Let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, let $\phi \in C^{0,1}(U;\mathbb{R}^n)$, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive Borel function. We have that*

$$\int_{\partial\Omega} g(\phi(\sigma))\, J^{\partial\Omega}\phi \, d\mathcal{H}^{n-1} = \int_{\phi(\partial\Omega)} g(\sigma)\, d\mathcal{H}^{n-1}.$$

Definition 2.22 (Tangential divergence)**.** Let Ω be a bounded open set of class C^1, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$ and let $\phi \in C^{0,1}(U;\mathbb{R}^n)$. We define the *tangential divergence of* ϕ as

$$\operatorname{div}^{\partial\Omega}\phi = \sum_{j=1}^{n-1} (D\phi\,\tau_j) \cdot \tau_j$$

Definition 2.23 (Mean Curvature)**.** Let Ω be a bounded open set with $C^{1,1}$ boundary and let

$$\nu : \partial\Omega \to \mathbb{S}^{n-1}$$

be the outer unit normal to its boundary. Let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$ and let $X \in C^{0,1}(U)$ be an extension of ν. For $\mathcal{H}^{n-1}$-a.e. $\sigma \in \partial\Omega$ we define the *mean curvature of* $\partial\Omega$ as

$$H(\sigma) = \operatorname{div}^{\partial\Omega} X = \sum_{i=1}^{n-1} k_i(\sigma).$$

Remark 2.24. Let Ω be a bounded open set of class $C^{1,1}$, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, let $X \in C^{0,1}(U;\mathbb{R}^n)$, and let $\phi(x) = x + tX(x)$. By direct computations, we have that

$$J^{\partial\Omega}\phi(\sigma) = 1 + t\,\mathrm{div}^{\partial\Omega} X(\sigma) + t^2 R(t,\sigma)$$

where the remainder R is a bounded function. In particular, if X is an extension of ν, we have

$$J^{\partial\Omega}\phi(\sigma) = 1 + tH(\sigma) + t^2 R(t,\sigma).$$

Theorem 2.25 (Coarea formula)**.** *Let $f \in C^{0,1}(\mathbb{R}^n)$, let $g \in L^1(\mathbb{R}^n)$, and let $U \subset \mathbb{R}^n$ be an open set. Then*

$$\int_U g(x)|\nabla f(x)|\,dx = \int_{\mathbb{R}}\int_{U\cap\{\,f=t\,\}} g(y)\,d\mathcal{H}^{n-1}(y)\,dt.$$

2.4 Γ-convergence

In this section, we recall some basic properties of the Γ-convergence and the asymptotic development by Γ-convergence. We refer, for instance, to [31] and [11] for the following notions.

Definition 2.26. Let X be a metric space and, for any $\varepsilon > 0$, let us consider the functionals $\mathcal{F}_\varepsilon, \mathcal{F}_0 : X \to \mathbb{R}\cup\{\,+\infty\,\}$. We will say that $\mathcal{F}_\varepsilon$ Γ-*converges, with respect to the strong topology in* X*, as* $\varepsilon \to 0^+$ to $\mathcal{F}_0$ if for every $x \in X$ the following conditions hold:

- for every sequence $\{\,x_\varepsilon\,\} \subset X$ converging to x,

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}_0(x);$$

- there exists a sequence $\{\,x_\varepsilon\,\} \subset X$ converging to x such that

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \leq \mathcal{F}_0(x).$$

In particular, from the definition, if $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}_0$, for every $x \in X$ there exists a recovery sequence $\{\,x_\varepsilon\,\} \subset X$, converging to x, such that

$$\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) = \mathcal{F}_0(x).$$

We have the following

Proposition 2.27. *Let X be a metric space and, for any $\varepsilon > 0$, let us consider the functionals $\mathcal{F}_\varepsilon, \mathcal{F}_0 : X \to \mathbb{R}\cup\{\,+\infty\,\}$ such that $\mathcal{F}_\varepsilon$ Γ-converges, with respect to the strong topology in X as $\varepsilon \to 0^+$ to $\mathcal{F}_0$. Let $\{\,x_\varepsilon\,\}$ be a sequence in X such that*

$$\mathcal{F}_\varepsilon(x_\varepsilon) = \min_X \mathcal{F}_\varepsilon.$$

If there exists $\overline{x} \in X$ such that x_ε converges to $\overline{x}$, then

$$\mathcal{F}_0(\overline{x}) = \min_X \mathcal{F}_0 = \lim_{\varepsilon\to 0^+} \min_X \mathcal{F}_\varepsilon.$$

Let

$$m_0 = \inf_X \mathcal{F}_0,$$

and, for every $x \in X$, let

$$\delta\mathcal{F}_\varepsilon(x) = \frac{\mathcal{F}_\varepsilon(x) - m_0}{\varepsilon}$$

Definition 2.28. If there exists a functional $\mathcal{F}^{(1)}\colon X \to \mathbb{R} \cup \{ +\infty \}$ such that $\delta\mathcal{F}_\varepsilon$ Γ-converges, with respect to the strong topology in X, as $\varepsilon \to 0^+$ to $\mathcal{F}^{(1)}$, we say that $\mathcal{F}^{(1)}$ is the *first-order asymptotic development by* Γ*-convergence* for the functional $\mathcal{F}_\varepsilon$.

Let

$$\mathcal{U}_0 = \{ x \in X \mid \mathcal{F}_0(x) = m_0 \} .$$

The interest in the previous definition is justified by the following

Remark 2.29. Let $\{ x_\varepsilon \}$ be a sequence in X such that

$$\mathcal{F}_\varepsilon(x_\varepsilon) = \min_X \mathcal{F}_\varepsilon,$$

and assume that there exists $\overline{x} \in X$ such that x_ε converges to $\overline{x}$; then, by Proposition 2.27, we have that $\overline{x} \in \mathcal{U}_0$ and

$$\mathcal{F}^{(1)}(\overline{x}) = \min_X \mathcal{F}^{(1)} = \lim_{\varepsilon \to 0^+} \frac{\mathcal{F}_\varepsilon(x_\varepsilon) - m_0}{\varepsilon}.$$

In particular, we have

$$\mathcal{F}_\varepsilon(x_\varepsilon) = m_0 + \varepsilon\mathcal{F}^{(1)}(\bar{x}) + o(\varepsilon).$$

Chapter 3

The Capacitary Problem

In this chapter, we will study the optimisation problem of finding the best configuration of insulating material surrounding a conductor kept at a fixed, homogeneous temperature. In particular, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded open set, representing the conductor, and let A be a set containing Ω, such that the set $\Sigma = A \setminus \overline{\Omega}$ represents the configuration of insulating material. If the conductor is kept at fixed temperature $u_A = 1$, the steady-state temperature of the configuration is given by

$$\begin{cases} u_A = 1 & \text{in } \overline{\Omega}, \\ \Delta u_A = 0 & \text{in } A \setminus \overline{\Omega}, \\ \partial_\nu u_A + \beta u_A = 0 & \text{on } \partial A, \end{cases} \tag{3.1}$$

where ν is the outer unit normal to the boundary of A, and β is a positive parameter. Then the best configuration is the one that minimises the energy

$$\mathcal{E}_c(A) = k \int_A |\nabla u_A|^2 \, dx + \beta \int_{\partial A} u_A^2 \, d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n\big(A \setminus \overline{\Omega}\big). \tag{3.2}$$

This problem has been introduced in the context of SBV functions in [28], where the authors proved the existence of a solution and the regularity of its jump set. An analogous problem has also been studied in [22].

This chapter is structured as follows.

In Section 3.1 we prove existence and regularity results for the minimiser of a non-linear version of the problem. In Section 3.2 we study the asymptotic behaviour of the problem in the thin layer setting, and characterise the solution to the limiting problem. In Section 3.3 we discuss other optimisation problems related to the minimisation of (3.2).

3.1 Bulk insulating layer: existence and regularity

The content of this section is based on the results of [1], which are a generalisation of the arguments of [28] to a non-linear setting.

Let $p, q > 1$ be two exponents, in analogy with the boundary value problem (3.2), consider the non-linear boundary value problem

$$\begin{cases} u_A = 1 & \text{in } \overline{\Omega}, \\ \Delta_p u_A = 0 & \text{in } A \setminus \overline{\Omega}, \\ |\nabla u_A|^{p-2} \partial_\nu u_A + q\beta |u_A|^{q-2} u_A = 0 & \text{on } \partial A, \end{cases} \tag{3.3}$$

where $\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big)$ is the *p-Laplace operator*. We aim to minimise the associated energy

$$\mathcal{E}_c(A, v) = k \int_A |\nabla v|^p \, dx + \beta \int_{\partial A} v^q \, d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n\big(A \setminus \overline{\Omega}\big),$$

where A is open set with Lipschitz boundary containing Ω and $v \in W^{1,p}(A)$ with $v = 1$ almost everywhere in Ω. We remark that for a fixed A, the function u_A solution to the non-linear boundary value problem (3.3), is the minimiser of $\mathcal{E}_c(A, \cdot)$. As the minimisation of the energy as a function of the couple (A, v) can be quite challenging, the idea of [28] is to identify the set A with the set $\{ v > 0 \}$ and to extend the function v to be equal to zero outside of A. Such an extension of the function v is in $\mathrm{SBV}(\mathbb{R}^n)$ and the energy can be written as

$$\mathcal{E}_c(v) = k \int_{\mathbb{R}^n} |\nabla v|^p \, dx + \beta \int_{J_v} (\underline{v}^q + \overline{v}^q) \, d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n(\{ v > 0 \} \setminus \Omega), \tag{3.4}$$

where ∇v is the absolutely continuous part of the distributional gradient of v, J_v is the jump set of v, and $\overline{v}$ and $\underline{v}$ are the approximate upper and lower limits respectively.

Then the minimisation problem is

$$\inf \left\{ \mathcal{E}_c(v) \;\middle|\; \begin{array}{c} v \in \mathrm{SBV}(\mathbb{R}^n), \\ v(x) = 1 \text{ in } \Omega \end{array} \right\}.$$

Notice that if $v \in \mathrm{SBV}(\mathbb{R}^n)$ with $v = 1$ almost everywhere in Ω, letting $v_0 = \max\{ 0, \min\{ v, 1 \} \}$ we have that v_0 is still admissible and $\mathcal{E}_c(v_0) \le \mathcal{E}_c(v)$ so it suffices to consider the problem

$$\inf \left\{ \mathcal{E}_c(v) \;\middle|\; \begin{array}{c} v \in \mathrm{SBV}(\mathbb{R}^n), \\ v(x) \in [0,1] \; \mathcal{L}^n\text{-a.e.}, \\ v(x) = 1 \text{ in } \Omega \end{array} \right\}. \tag{3.5}$$

For simplicity's sake, we will assume $k = 1 = C_0$.

In the following theorems, we summarise the main results of this section.

Theorem 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $p, q > 1$ be exponents satisfying one of the following conditions:*

- $1 < p < n$, *and* $1 < q < \dfrac{p(n-1)}{n-p} := p_*$;
- $n \le p < \infty$, *and* $1 < q < \infty$.

Then there exists a solution u to problem (3.5) *and there exists a constant $\delta_0 = \delta_0(\Omega, \beta, p, q) > 0$ such that*

$$u > \delta_0 \tag{3.6}$$

$\mathcal{L}^n$-almost everywhere in $\{ u > 0 \}$, and there exists $\rho(\delta_0) > 0$ such that

$$\operatorname{supp} u \subseteq B_{\rho(\delta_0)}.$$

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $p, q > 1$ be exponents satisfying the assumptions of Theorem 3.1. Then there exist positive constants $C(\Omega, \beta, p, q)$, $c(\Omega, \beta, p, q)$, $C_1(\Omega, \beta, p, q)$ such that if u is a minimiser to problem* (3.5)*, then*

$$c\, r^{n-1} \le \mathcal{H}^{n-1}(J_u \cap B_r(x)) \le C\, r^{n-1},$$

and

$$\mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \ge C_1\, r^n,$$

for every $x \in K_u$ with $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$, where we recall that K_u is the closure of J_u.

In particular, this implies the essential closedness of the jump set J_u outside of Ω, namely

$$\mathcal{H}^{n-1}((K_u \setminus J_u) \setminus \bar{\Omega}) = 0.$$

In subsection 3.1.1, we prove that the a priori estimate (3.6) holds for inward minimisers (see Definition 3.4), such an estimate will be crucial in the proof of Theorem 3.1 in subsection 3.1.2. Finally, in subsection 3.1.3 we prove Theorem 3.2.

Remark 3.3. Notice that the condition on the exponents is undoubtedly verified when $p \geq q > 1$. Furthermore, if Ω is a set with Lipschitz boundary, the exponent p_* is the optimal exponent such that

$$W^{1,p}(\Omega) \subset\subset L^q(\partial\Omega) \qquad \forall q \in [1, p_*).$$

3.1.1 Lower Bound

In the following, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and that p and q are two positive real numbers such that

$$\frac{q'}{p'} > 1 - \frac{1}{n} \tag{3.7}$$

where p' and q' are the Hölder conjugates of p and q respectively.

The following definition, introduced in [28], will be crucial in proving the existence of a solution.

Definition 3.4. Let $v \in \mathrm{SBV}(\mathbb{R}^n)$ be a function such that $v = 1$ a.e. in Ω. We say that v is an *inward minimiser* if

$$\mathcal{E}_c(v) \leq \mathcal{E}_c(v\chi_A),$$

for every set of finite perimeter A containing Ω, where χ_A is the characteristic function of set A.

Let $A \subset \mathbb{R}^n$ be a set of finite perimeter such that $\Omega \subset A$, and let $v \in \mathrm{SBV}(\mathbb{R}^n)$. We will make use of the following expression

$$\begin{aligned}\mathcal{E}_c(v\chi_A) = &\int_A |\nabla v|^p \, dx + \beta \int_{J_v \cap A^{(1)}} (\underline{v}^q + \overline{v}^q) \, d\mathcal{H}^{n-1} + \beta \int_{\partial^* A \setminus J_v} v^q \, d\mathcal{H}^{n-1} \\ &+ \beta \int_{J_v \cap \partial^* A} \gamma^-_{\partial A}(v)^q \, d\mathcal{H}^{n-1} + \mathcal{L}^n((\{\, v > 0 \,\} \cap A) \setminus \Omega),\end{aligned} \tag{3.8}$$

Let B be a ball containing Ω, then $\chi_B \in \mathrm{SBV}(\mathbb{R}^n)$ and $\chi_B = 1$ in Ω, we will denote $\mathcal{E}_c(\chi_B)$ by $\tilde{\mathcal{E}}_c$.

Theorem 3.5. *There exists a positive constant $\delta = \delta(\Omega, \beta, p, q)$ such that if u is an inward minimiser with $\mathcal{E}_c(u) \leq 2\tilde{\mathcal{E}}_c$, then*

$$u > \delta$$

$\mathcal{L}^n$-almost everywhere in $\{\, u > 0 \,\}$.

Proof. Let $0 < t < 1$ and

$$g(t) = \int_{\{\, u \leq t \,\} \setminus J_u} u^{q-1} |\nabla u| \, dx = \int_0^t s^{q-1} P(\{\, u > s \,\}; \mathbb{R}^n \setminus J_u) \, ds.$$

For every such t, we have

$$g(t) \leq \left(\int_{\{\, u \leq t \,\}} u^{(q-1)p'} \, dx \right)^{\frac{1}{p'}} \left(\int_{\{\, u \leq t \,\} \setminus J_u} |\nabla u|^p \, dx \right)^{\frac{1}{p}} \leq \mathcal{E}_c(u) \leq 2\tilde{\mathcal{E}}_c. \tag{3.9}$$

Let $u_t = u\chi_{\{\, u > t \,\}}$. Using (3.8) we have

$$\begin{aligned}0 \leq &\mathcal{E}_c(u_t) - \mathcal{E}_c(u) \\ = &\beta \int_{\partial^* \{\, u > t \,\} \setminus J_u} \overline{u}^q \, d\mathcal{H}^{n-1} - \int_{\{\, u \leq t \,\} \setminus J_u} |\nabla u|^p \, dx - \beta \int_{J_u \cap \partial^* \{\, u > t \,\}} \underline{u}^q \, d\mathcal{H}^{n-1} + \\ &- \beta \int_{J_u \cap \{\, u > t \,\}^{(0)}} (\overline{u}^q + \underline{u}^q) \, d\mathcal{H}^{n-1} - \mathcal{L}^n(\{\, 0 < u \leq t \,\}),\end{aligned}$$

and rearranging the terms,

$$\begin{aligned}\int_{\{u\le t\}\setminus J_u} |\nabla u|^p\,dx + \beta\int_{J_u\cap\partial^*\{u>t\}} \underline{u}^q\,d\mathcal{H}^{n-1} + \beta\int_{J_u\cap\{u>t\}^{(0)}} (\overline{u}^q+\underline{u}^q)\,d\mathcal{H}^{n-1} + \\ + \mathcal{L}^n(\{0<u\le t\}) \le \beta t^q P(\{u>t\};\mathbb{R}^n\setminus J_u) = \beta t g'(t).\end{aligned} \tag{3.10}$$

On the other hand,

$$\begin{aligned} g(t) &= \int_{\{u\le t\}\setminus J_u} u^{q-1}|\nabla u|\,dx \\ &\le \left(\int_{\{u\le t\}} u^{(q-1)p'}\,dx\right)^{\frac{1}{p'}}\left(\int_{\{u\le t\}\setminus J_u}|\nabla u|^p\,dx\right)^{\frac1p} \\ &\le \left(\mathcal{L}^n(\{0<u\le t\})\right)^{\frac{1}{p'\gamma'}}\left(\int_{\{u\le t\}} u^{q1^*}\,dx\right)^{\frac{1}{q'1^*}}\left(\int_{\{u\le t\}\setminus J_u}|\nabla u|^p\,dx\right)^{\frac1p},\end{aligned}$$

where we used

$$1^* = \frac{n}{n-1}, \qquad \text{and} \qquad \gamma = \frac{q1^*}{(q-1)p'},$$

and $\gamma > 1$ by (3.7). By classical BV embedding in L^{1^*} applied to the function $(u\chi_{\{u\le t\}})^q$ and the estimate (3.10), we have

$$g(t) \le C(n,\beta)\Big(tg'(t)\Big)^{1-\frac{n-1}{q'n}}\left(\int_{\mathbb{R}^n} d\big|D(u\chi_{\{u\le t\}})^q\big|\right)^{\frac{1}{q'}}.$$

We can compute

$$\begin{aligned}\int_{\mathbb{R}^n} d\big|D(u\chi_{\{u\le t\}})^q\big| \le q\left(\mathcal{L}^n(\{0<u\le t\})\right)^{\frac{1}{p'}}\left(\int_{\{u\le t\}\setminus J_u}|\nabla u|^p\,dx\right)^{\frac1p} + \\ + \int_{J_u\cap\{u>t\}^{(0)}}(\overline{u}^q+\underline{u}^q)\,d\mathcal{H}^{n-1} + \int_{J_u\cap\partial^*\{u>t\}}\underline{u}^q\,d\mathcal{H}^{n-1} + \\ + t^qP(\{u>t\};\mathbb{R}^n\setminus J_u) \le (2+q\beta)tg'(t).\end{aligned}$$

We therefore get

$$g(t) \le C(n,\beta,q)\big(tg'(t)\big)^{1+\frac{1}{nq'}}.$$

Let $0<t_0<1$ such that $g(t_0)>0$, then for every $t_0<t<1$, we have $g(t)>0$ and

$$\frac{g'(t)}{g(t)^{\frac{nq}{q(n+1)-1}}} \ge \frac{C(n,\beta,q)}{t},$$

integrating from t_0 to 1, we have

$$g(1)^{\frac{q-1}{q(n+1)-1}} - g(t_0)^{\frac{q-1}{q(n+1)-1}} \ge C(n,\beta,q)\log\frac{1}{t_0},$$

so that, using (3.9),

$$g(t_0)^{\frac{q-1}{q(n+1)-1}} \le (2\tilde{\mathcal{E}}_c)^{\frac{q-1}{q(n+1)-1}} + C(n,\beta,q)\log t_0.$$

Let

$$\delta = \exp\left(-\frac{(2\tilde{\mathcal{E}}_c)^{\frac{q-1}{q(n+1)-1}}}{C(n,\beta,q)}\right),$$

for every $t_0<\delta$ we would have $g(t_0)<0$, which is a contradiction. Therefore $g(t)=0$ for every $t<\delta$, from which $u>\delta$ $\mathcal{L}^n$-almost everywhere on $\{u>0\}$. $\square$

Remark 3.6. From Theorem 3.5, if u is an inward minimiser with $\mathcal{E}_c(u) \leq 2\tilde{\mathcal{E}}_c$, we have that

$$\partial^* \{ u > 0 \} \subseteq J_u \subseteq K_u.$$

Indeed, on $\partial^* \{ u > 0 \}$ we have that, by definition, $\underline{u} = 0$ and that, since $u \geq \delta$ $\mathcal{L}^n$-almost everywhere in $\{ u > 0 \}$, $\overline{u} \geq \delta$.

Proposition 3.7. *There exists a positive constant $\delta_0 = \delta_0(\Omega, \beta, p, q) < \delta$ such that if u is an inward minimiser with $\mathcal{E}_c(u) \leq 2\tilde{\mathcal{E}}_c$, then u is supported on $B_{\rho(\delta_0)}$, where $\rho(\delta_0) = \delta_0^{1-q}$ and $B_{\rho(\delta_0)}$ is the ball centred at the origin with radius $\rho(\delta_0)$. Moreover there exist positive constants $C(\Omega, \beta, p, q), C_1(\Omega, \beta, p, q)$ such that, for any $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$ we have*

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C(\Omega, p, q) r^{n-1}, \tag{3.11}$$

and if $x \in K_u$, then

$$\mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq C_1(\Omega, p, q) r^n. \tag{3.12}$$

Proof. By Theorem 3.5, if u is an inward minimiser, we have

$$\int_{J_u \cap B_r(x)} (\overline{u}^q + \underline{u}^q)\, d\mathcal{H}^{n-1} \geq \delta^q \mathcal{H}^{n-1}(J_u \cap B_r(x)),$$

on the other hand, using $u\chi_{\mathbb{R}^n \setminus B_r(x)}$ as a competitor for u, we have

$$\int_{J_u \cap B_r(x)} (\overline{u}^q + \underline{u}^q)\, d\mathcal{H}^{n-1} \leq \int_{\partial B_r(x) \cap \{ u > 0 \}^{(1)}} (\overline{u}^q + \underline{u}^q)\, d\mathcal{H}^{n-1} \leq C(n) r^{n-1}.$$

Let now $x \in K_u$ and consider $\mu(r) = \mathcal{L}^n\Big(B_r(x) \cap \{ u > 0 \}^{(1)}\Big)$. Using the isoperimetric inequality and inequality (3.11), we have that for almost every $r \in (0, d(x, \Omega))$

$$\begin{aligned} 0 < \mu(r) &\leq K(n)\, P\Big(B_r(x) \cap \{ u > 0 \}^{(1)}\Big)^{\frac{n}{n-1}} \\ &\leq K(\Omega, \beta, p, q)\, P\Big(B_r(x); \{ u > 0 \}^{(1)}\Big)^{\frac{n}{n-1}}. \end{aligned}$$

Notice that we used Remark 3.6 in the last inequality. We have

$$\mu(r) \leq K \mu'(r)^{\frac{n}{n-1}}.$$

Integrating the differential inequality, we obtain

$$\mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq C_1(\Omega, \beta, p, q) r^n.$$

Finally, let $\delta_0 > 0$ and $x \in K_u$ such that $d(x, \Omega) > \rho(\delta_0) = \delta_0^{1-q}$. By (3.12)

$$C_1(\Omega, \beta, p, q) \rho(\delta_0)^n \leq \mathcal{L}^n(\{ u > 0 \} \cap \Omega) \leq 2\tilde{\mathcal{E}}_c,$$

which leads to a contradiction if δ_0 is too small, hence there exists a positive value $\delta_0(\Omega, \beta, p, q)$ such that $\{ u > 0 \} \subset B_{\rho(\delta_0)}$. ☐

3.1.2 Existence

In this section, we are going to prove the existence of a solution u to the problem (3.5). Let us denote

$$H_a = \left\{ u \in \mathrm{SBV}(\mathbb{R}^n) \;\middle|\; \begin{array}{c} u(x) = 1 \text{ in } \Omega \\ u(x) \in \{ 0 \} \cup [a, 1] \ \mathcal{L}^n\text{-a.e.} \\ \operatorname{supp} u \subseteq B_{\frac{1}{a^{q-1}}} \end{array} \right\}.$$

We also denote by H_0 the set

$$H_0 = \left\{ u \in \mathrm{SBV}(\mathbb{R}^n) \;\middle|\; \begin{array}{c} u(x) = 1 \text{ in } \Omega \\ u(x) \in [0,1] \ \mathcal{L}^n\text{-a.e.} \end{array} \right\}.$$

Notice that if $u \in H_0$ is an inward minimiser, by Theorem 3.5 and Proposition 3.7, then $u \in H_{\delta_0}$.

Proposition 3.8. *Let $u \in H_0$. Then u is a minimiser for the functional* (3.4) *on H_0 if and only if $u \in H_{\delta_0}$ and*

$$\mathcal{E}_c(u) \le \mathcal{E}_c(v) \qquad \forall v \in H_{\delta_0}.$$

Proof. As we observed before, if u is a minimiser over H_0 then u is in H_{δ_0}, hence it is a minimiser over H_{δ_0}. Conversely, let us take $u \in H_{\delta_0}$ a minimiser over H_{δ_0}, and let us consider in addition $v \in H_0$. Without loss of generality assume $\mathcal{E}_c(v) \le 2\tilde{\mathcal{E}}_c$. We will prove that there exists a sequence w_k of inward minimisers such that

$$\mathcal{E}_c(w_k) \le \mathcal{E}_c(v) + \frac{C}{k^{q-1}}.$$

We first construct a family of functions $v_a \in H_a$ such that

$$\mathcal{E}_c(v_a) \le \mathcal{E}_c(v) + r(a),$$

with $\lim_{a\to 0} r(a) = 0$. Let $0 < a < 1$, and let $v_a = v\chi_{\{v \ge a\} \cap B_{\rho(a)}}$, where $\rho(a) = a^{1-q}$, we have

$$\begin{aligned} \mathcal{E}_c(v_a) - \mathcal{E}_c(v) &\le \beta \int_{\partial^*(\{v\ge a\}\cap B_{\rho(a)})\setminus J_v} v^q \, d\mathcal{H}^{n-1} \\ &\le \beta a^q P(\{v \ge a\}) + \beta \int_{(\partial B_{\rho(a)} \cap \{v \ge a\})\setminus J_v} v^q \, d\mathcal{H}^{n-1} \\ &\le \beta a^q \left(P(\{v \ge a\}) + \frac{1}{a^q} \int_{(\partial B_{\rho(a)} \cap \{v \ge a\})\setminus J_v} v \, d\mathcal{H}^{n-1} \right). \end{aligned} \tag{3.13}$$

In order to estimate the right-hand side, fix $t \in (0,1)$, and observe that by the coarea formula

$$\int_0^t P(\{v \ge a\}) \, da \le |Dv|(\mathbb{R}^n), \tag{3.14}$$

while with a change of variables,

$$\int_0^t \frac{1}{a^q} \int_{(\partial B_{\rho(a)} \cap \{v \ge a\})\setminus J_v} v \, d\mathcal{H}^{n-1} \, da \le (q-1) \int_0^{+\infty} \int_{\partial B_r \setminus J_v} v \, d\mathcal{H}^{n-1} \, dr = (q-1)\|v\|_{L^1(\mathbb{R}^n)}.$$

$$\int_0^t \left(P(\{v \ge a\}) + \frac{1}{a^q} \int_{(\partial B_{\rho(a)} \cap \{v \ge a\})\setminus J_v} v \, d\mathcal{H}^{n-1} \right) da \le q\|v\|_{\mathrm{BV}}.$$

By mean value theorem, for every $k \in \mathbb{N}$ we can find $a_k \le 1/k$ such that

$$P(\{v \ge a_k\}) + \frac{1}{a_k^q} \int_{(\partial B_{\rho(a_k)} \cap \{v \ge a_k\})\setminus J_v} v \, d\mathcal{H}^{n-1} \le \frac{q\|v\|_{\mathrm{BV}}}{a_k},$$

and in (3.13) we get

$$\mathcal{E}_c(v_{a_k}) \le \mathcal{E}_c(v) + q\beta a_k^{q-1}\|v\|_{\mathrm{BV}} \le \mathcal{E}_c(v) + q\beta \frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

We now construct the aforementioned sequence of inward minimisers: let us consider the functional

$$\mathcal{F}_k(A) = \mathcal{E}_c(v_{a_k}\chi_A),$$

with A containing Ω and contained in $\{ v_{a_k} > 0 \}$. If we consider A_j a minimising sequence for $\mathcal{F}_k$, then they are certainly equibounded. Moreover,

$$\begin{aligned}\mathcal{F}_k(A_j) &\geq \mathcal{L}^n(A_j \setminus \Omega) + \beta \int_{J_{\chi_{A_j} v_{a_k}}} \left(\underline{\chi_{A_j} v_{a_k}}^q + \overline{\chi_{A_j} v_{a_k}}^q \right) d\mathcal{H}^{n-1} \\ &\geq \mathcal{L}^n(A_j) + \beta a_k^q \mathcal{H}^{n-1}(J_{\chi_{A_j} v_{a_k}}) - \mathcal{L}^n(\Omega).\end{aligned}$$

Notice in addition that since $v_{a_k} \geq a_k$ on its support, then the jump set $J_{\chi_{A_j} v_{a_k}}$ clearly contains $\partial^* A_j$. We now have that χ_{A_j} satisfies the conditions of Theorem 2.12, and eventually extracting a subsequence we can suppose that

$$A_j \xrightarrow{L^1} A^{(k)},$$

with a suitable $A^{(k)}$, and moreover, letting $w_k = \chi_{A^{(k)}} v_{a_k}$, we have

$$\mathcal{E}_c(w_k) \leq \inf_{\Omega \subseteq A \subseteq \{ v_{a_k} > 0 \}} \mathcal{F}_k(A) \leq \mathcal{E}_c(v_{a_k}) \leq \mathcal{E}_c(v) + q\beta \frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

By construction w_k is an inward minimiser, therefore we have $w_k \in H_{\delta_0}$, and consequently, we can compare it with u, obtaining

$$\mathcal{E}_c(u) \leq \mathcal{E}_c(w_k) \leq \mathcal{E}_c(v) + q\beta \frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

Letting k go to infinity, we get the thesis. □

Proposition 3.9. *There exists a minimiser for problem* (3.5)*.*

Proof. By Proposition 3.8 and Theorem 3.5, it is enough to find a minimiser in H_{δ_0}. Let u_k be a minimising sequence in H_{δ_0}, then, for k large enough, we have

$$\beta \delta_0^q \mathcal{H}^{n-1}(J_{u_k}) + \int_{\mathbb{R}^n} |\nabla u_k|^p \, dx \leq \mathcal{E}_c(u_k) \leq 2\tilde{\mathcal{E}}_c.$$

From Theorem 2.12 we have that there exists $u \in H_{\delta_0}$ such that, up to a subsequence, u_k converges to u in L^1_{loc} and

$$\mathcal{E}_c(u) \leq \liminf_k \mathcal{E}_c(u_k),$$

therefore u is a solution. □

Proof of Theorem 3.1. The result is obtained by joining Proposition 3.9 and Theorem 3.5. □

3.1.3 Density estimates

In this section, we prove the density estimates in Theorem 3.2 by adapting techniques used in [28], analogous to classical ones used in [44] to prove density estimates for the jump set of almost-quasi minimisers of the Mumford-Shah functional.

Definition 3.10. Let $u \in \mathrm{SBV}(\mathbb{R}^n)$ be a function such that $u = 1$ a.e. in Ω. We say that u is a *local minimiser* for $\mathcal{E}_c$ on a set of finite perimeter $E \subset \mathbb{R}^n \setminus \Omega$, if

$$\mathcal{E}_c(u) \leq \mathcal{E}_c(v),$$

for every $v \in \mathrm{SBV}(\mathbb{R}^n)$ such that $u - v$ has support in E.

Let E be a set of finite perimeter. We introduce the notation

$$\mathcal{E}_c(u;E) = \int_E |\nabla u|^p \, dx + \beta \int_{J_u \cap E} (\overline{u}^q + \underline{u}^q) \, d\mathcal{H}^{n-1} + \mathcal{L}^n(\{ u > 0 \} \cap E).$$

To prove Theorem 3.2, we will use the following Poincaré-Wirtinger type inequality, whose proof can be found in [44, Theorem 3.1 and Remark 3.3]. Let γ_n be the isoperimetric constant relative to the balls of $\mathbb{R}^n$, i.e.

$$\min \left\{ \mathcal{L}^n(E \cap B_r)^{\frac{n-1}{n}}, \mathcal{L}^n(E \setminus B_r)^{\frac{n-1}{n}} \right\} \leq \gamma_n P(E; B_r),$$

for every Borel set E, then

Proposition 3.11. *Let $p \geq 1$ and let $u \in \mathrm{SBV}(B_r)$ such that*

$$\big(2\gamma_n \mathcal{H}^{n-1}(J_u \cap B_r)\big)^{\frac{n}{n-1}} < \frac{\mathcal{L}^n(B_r)}{2}. \tag{3.15}$$

Then there exist numbers $-\infty < \tau^- \leq m \leq \tau^+ < +\infty$ such that the function

$$\tilde{u} = \max \{ \min \{ u, \tau^+ \}, \tau^- \},$$

satisfies

$$\|\tilde{u} - m\|_{L^p} \leq C \|\nabla u\|_{L^p}$$

and

$$\mathcal{L}^n(\{ u \neq \tilde{u} \}) \leq C \big(\mathcal{H}^{n-1}(J_u \cap B_r)\big)^{\frac{n}{n-1}},$$

where the constants depend only on n, p, and r.

Lemma 3.12. *Let $u \in H_s$ be a local minimiser on $B_r(x)$ in the sense of Definition 3.10. For sufficiently small values of τ, there exist values r_0, ε_0 depending only on n, τ, β, p, q and s such that, if $r < r_0$,*

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq \varepsilon_0 r^{n-1},$$

and

$$\mathcal{E}_c(u; B_r(x)) \geq r^{n-\frac{1}{2}},$$

then

$$\mathcal{E}_c(u; B_{\tau r}(x)) \leq \tau^{n-\frac{1}{2}} \mathcal{E}_c(u; B_r(x)).$$

Proof. Without loss of generality, assume $x = 0$. Assume by contradiction that the conclusion fails, then for every $\tau > 0$ there exists a sequence $u_k \in H_s$ of local minimisers on B_{r_k}, with $\lim_k r_k = 0$, such that

$$\frac{\mathcal{H}^{n-1}(J_{u_k} \cap B_{r_k})}{r_k^{n-1}} = \varepsilon_k,$$

with $\lim_k \varepsilon_k = 0$,

$$\mathcal{E}_c(u_k; B_{r_k}) \geq r_k^{n-\frac{1}{2}}, \tag{3.16}$$

and yet

$$\mathcal{E}_c(u_k; B_{\tau r_k}) > \tau^{n-\frac{1}{2}} \mathcal{E}_c(u_k; B_{r_k}). \tag{3.17}$$

For every $t \in [0,1]$, we define the sequence of monotone functions

$$\alpha_k(t) = \frac{\mathcal{E}_c(u_k; B_{t r_k})}{\mathcal{E}_c(u_k, B_{r_k})} \leq 1.$$

By compactness of $\mathrm{BV}([0,1])$ in $L^1([0,1])$, we can assume that, up to a subsequence, α_k converges $\mathcal{L}^1$-almost everywhere to a monotone function α. Moreover, notice that, by (3.17), for every k

$$\alpha_k(\tau) > \tau^{n-\frac{1}{2}}. \tag{3.18}$$

Our final aim is to prove that there exists a p-harmonic function $v \in W^{1,p}(B_1)$ such that for every t

$$\lim_{k\to+\infty} \alpha_k(t) = \alpha(t) = \int_{B_t} |\nabla v|^p \, dx.$$

Let

$$E_k = r_k^{p-n} \mathcal{E}_c(u_k; B_{r_k}), \qquad v_k(x) = \frac{u_k(r_k x)}{E_k^{1/p}}.$$

Then $v_k \in \mathrm{SBV}(B_1)$, and

$$\int_{B_1} |\nabla v_k|^p \, dx \le 1, \qquad \mathcal{H}^{n-1}(J_{v_k} \cap B_1) = \varepsilon_k.$$

Thus, applying the Poincaré-Wirtinger type inequality in Proposition 3.11 to functions v_k we obtain truncated functions $\tilde{v}_k$ and values m_k, such that

$$\int_{B_1} |\tilde{v}_k - m_k|^p \, dx \le C$$

and

$$\mathcal{L}^n(\{ v_k \neq \tilde{v}_k \}) \le C\big(\mathcal{H}^{n-1}(J_{v_k} \cap B_1)\big)^{\frac{n}{n-1}} \le C\varepsilon_k^{\frac{n}{n-1}}. \tag{3.19}$$

We prove that there exists $v \in W^{1,p}(B_1)$ such that

$$\tilde{v}_k - m_k \xrightarrow{L^p(B_1)} v,$$

$$\int_{B_\rho} |\nabla v|^p \, dx \le \alpha(\rho), \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1, \tag{3.20}$$

and

$$\lim_k \frac{r_k^{p-1}}{E_k} \mathcal{H}^{n-1}(\{ v_k \neq \tilde{v}_k \} \cap \partial B_\rho) = 0, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1. \tag{3.21}$$

Notice that

$$\int_{B_1} |\nabla(\tilde{v}_k - m_k)|^p \, dx \le \int_{B_1} |\nabla v_k|^p \, dx \le 1,$$

since $\tilde{v}_k$ is a truncation of v. From compactness theorems in SBV (see for instance [44, Theorem 3.5]), we have that $\tilde{v}_k - m_k$ converges in $L^p(B_1)$ and $\mathcal{L}^n$-almost everywhere to a function $v \in W^{1,p}(B_1)$, since $\mathcal{H}^{n-1}(J_{\tilde{v}_k})$ goes to 0 as $k \to +\infty$. Moreover, for every $\rho < 1$,

$$\int_{B_\rho} |\nabla v|^p \, dx \le \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, dx,$$

and

$$\int_{B_\rho} |\nabla v|^p \, dx \le \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, dx \le \liminf_k \alpha_k(\rho) = \alpha(\rho),$$

since by definition

$$\int_{B_\rho} |\nabla v_k|^p \, dx = \frac{r_k^{p-n}}{E_k} \int_{B_{\rho r_k}} |\nabla u_k|^p \, dx \le \frac{r_k^{p-n}}{E_k} \mathcal{E}_c(u_k; B_{\rho r_k}) \le \alpha_k(\rho).$$

Finally, up to a subsequence,

$$\lim_k \frac{r_k^{p-1}}{E_k}\mathcal{L}^n(\{\, v_k \neq \tilde{v}_k \,\}) = 0.$$

Indeed, by (3.19),

$$\frac{r_k^{p-1}}{E_k}\mathcal{L}^n(\{\, v_k \neq \tilde{v}_k \,\}) \le C\frac{r_k^{p-1}}{E_k}\varepsilon_k^{\frac{n}{n-1}},$$

which tends to zero as long as r_k^{p-1}/E_k is bounded. On the other hand, if r_k^{p-1}/E_k diverges, we could use the fact that $\varepsilon_k \le s^{-q}\mathcal{E}_c(u_k; B_{r_k})r_k^{1-n}$ and get

$$\frac{r_k^{p-1}}{E_k}\mathcal{L}^n(\{\, v_k \neq \tilde{v}_k \,\}) \le C\frac{r_k^{p-1}}{E_k}\left(\frac{E_k}{r_k^{p-1}}\right)^{\frac{n}{n-1}}$$

which goes to zero. Using Fubini's theorem, we have (3.21).

Let $\tilde{u}_k(x) = E_k^{1/p}\tilde{v}_k(\frac{x}{r_k})$, and for every $t \in [0,1]$ we define

$$\tilde{\alpha}_k(t) = \frac{\mathcal{E}_c(\tilde{u}_k; B_{tr_k})}{\mathcal{E}_c(u_k, B_{r_k})}.$$

The $\tilde{\alpha}_k$ functions are also monotone and bounded: the jump set of $\tilde{u}_k$ is contained in J_{u_k}, therefore we can write

$$\tilde{\alpha}_k(t) \le \alpha_k(t) + \frac{2\beta\mathcal{H}^{n-1}(J_{u_k}\cap B_{tr_k})}{\mathcal{E}_c(u_k; B_{r_k})} \le \left(1+\frac{2}{s^q}\right)\alpha_k(t),$$

using the fact that $u_k \in H_s$. As done for α_k, we can assume that $\tilde{\alpha}_k$ converges $\mathcal{L}^1$-almost everywhere to a function $\tilde{\alpha}$.

Let $I \subset [0,1]$ be the set of values ρ for which (3.21) holds, α_k and $\tilde{\alpha}_k$ converge and α and $\tilde{\alpha}$ are continuous. Notice that $\mathcal{L}^1(I) = 1$. Fix $\rho, \rho' \in I$ with $\rho < \rho' < 1$ and let

$$\mathcal{I}_k(\xi) = \beta E_k^{q/p-1}r_k^{p-1}\int_{J_\xi\cap(B_{\rho'}\setminus B_\rho)}\left(\overline{\xi}^q + \underline{\xi}^q\right)d\mathcal{H}^{n-1},$$

with $\xi \in \mathrm{SBV}(B_1)$. Let $w \in W^{1,p}(B_1)$ and consider η a smooth cut-off function supported on $B_{\rho'}$ and identically equal to 1 in B_ρ. Let

$$\varphi_k = ((w+m_k)\eta + \tilde{v}_k(1-\eta))\chi_{B_{\rho'}} + v_k\chi_{B_1\setminus B_{\rho'}}.$$

We want to prove that

$$\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) \ge \int_{B_{\rho'}\setminus B_\rho}|\nabla\tilde{v}_k|^p\,dx + \mathcal{I}_k(\tilde{v}_k), \tag{3.22}$$

and

$$\alpha_k(\rho') \le R_k + \int_{B_{\rho'}}|\nabla\varphi_k|^p\,dx + \mathcal{I}_k(\varphi_k), \tag{3.23}$$

where R_k goes to zero as k goes to infinity. We immediately compute

$$\begin{aligned}\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) =& \mathcal{E}_c(u_k; B_{r_k})^{-1}\left[\int_{B_{\rho' r_k}\cap B_{\rho r_k}}|\nabla\tilde{u}_k|^p\,dx + \beta\int_{J_{\tilde{u}_k}\cap(B_{\rho' r_k}\setminus B_{\rho r_k})}\left(\overline{\tilde{u}_k}^q + \underline{\tilde{u}_k}^q\right)d\mathcal{H}^{n-1}\right]\\ &+ \mathcal{E}_c(u_k; B_{r_k})^{-1}\mathcal{L}^n(\{\, \tilde{u}_k > 0 \,\}\cap(B_{\rho' r_k}\setminus B_{\rho r_k}))\\ \ge& \int_{B_{\rho'}\setminus B_\rho}|\nabla\tilde{v}_k|^p\,dx + E_k^{q/p-1}r_k^{p-1}\beta\int_{J_{\tilde{v}_k}\cap(B_{\rho'}\setminus B_\rho)}\left(\overline{\tilde{v}_k}^q + \underline{\tilde{v}_k}^q\right)d\mathcal{H}^{n-1},\end{aligned}$$

and then we have (3.22). Now let $\psi_k = E_k^{1/p}\varphi_k(x/r_k)$ and observe that ψ_k coincides with u_k outside $B_{\rho' r_k}$. We get from the local minimality of u_k that

$$\begin{aligned}\mathcal{E}_c(u_k;B_{r_k}) \le \mathcal{E}_c(\psi_k;B_{r_k}) &= \mathcal{E}_c(\psi_k;B_{\rho' r_k}) + \beta\int_{\{u_k\neq\tilde u_k\}\cap\partial B_{\rho' r_k}}\left(\underline{\psi_k}^q + \overline{\psi_k}^q\right)d\mathcal{H}^{n-1}\\ &\quad + \mathcal{E}_c(u_k;B_{r_k}\setminus\overline{B_{\rho' r_k}})\\ &\le \mathcal{E}_c(\psi_k;B_{\rho' r_k}) + 2\beta r_k^{n-1}\mathcal{H}^{n-1}(\{v_k\neq\tilde v_k\}\cap\partial B_{\rho'})\\ &\quad + \mathcal{E}_c(u_k;B_{r_k}\setminus\overline{B_{\rho' r_k}}).\end{aligned}\tag{3.24}$$

So, in particular, we have

$$\begin{aligned}\mathcal{E}_c(u_k;B_{\rho' r_k}) =&\mathcal{E}_c(u_k;B_{r_k}) - \mathcal{E}_c(u_k;B_{r_k}\setminus\overline{B_{\rho' rfcfc_k}}) - \beta\int_{J_{u_k}\cap\partial B_{\rho' r_k}}\left(\overline{u_k}^q + \underline{u_k}^q\right)d\mathcal{H}^{n-1}\\ \le& 2\beta r_k^{n-1}\mathcal{H}^{n-1}(\{v_k\neq\tilde v_k\}\cap\partial B_{\rho'}) + \mathcal{E}_c(\psi_k;B_{\rho' r_k}).\end{aligned}$$

Dividing by $\mathcal{E}_c(u_k;B_{r_k})$ and using (3.21) we get

$$\alpha_k(\rho') \le R_k + r_k^{p-n}E_k^{-1}\mathcal{E}_c(\psi_k;B_{\rho' r_k}).$$

With appropriate rescaling, we have

$$r_k^{p-n}E_k^{-1}\mathcal{E}_c(\psi_k;B_{\rho' r_k}) = \int_{B_{\rho'}}|\nabla\varphi_k|^p\,dx + \mathcal{I}_k(\varphi_k) + r_k^pE_k^{-1}\mathcal{L}^n(\{\varphi_k>0\}\cap B_{\rho'}).$$

From (3.16) and the definition of E_k, we have

$$r_k^pE_k^{-1}\mathcal{L}^n(\{\varphi_k>0\}\cap B_{\rho'}) \le \omega_n r_k^{1/2},$$

and then we get (3.23).

We want to prove that for every $\varphi\in W^{1,p}(B_1)$ such that $v-\varphi$ is supported on B_ρ, we have

$$\alpha(\rho') \le \int_{B_\rho}|\nabla\varphi|^p\,dx + C\big[\tilde\alpha(\rho') - \tilde\alpha(\rho)\big] + C\int_{B_{\rho'}\setminus B_\rho}|\nabla\varphi|^p\,dx,\tag{3.25}$$

where C does not depend on either ρ or ρ'. From the definition of φ_k, we have that on B_ρ

$$\nabla\varphi_k = \nabla w$$

and on $B_{\rho'}\setminus B_\rho$

$$\nabla\varphi_k = \eta\nabla w + (w+m_k-\tilde v_k)\nabla\eta + \nabla\tilde v_k(1-\eta),$$

so that

$$\begin{aligned}\int_{B_{\rho'}}|\nabla\varphi_k|^p\,dx \le& \int_{B_\rho}|\nabla w|^p\,dx\\ &+ C\left[\int_{B_{\rho'}\setminus B_\rho}|\nabla\tilde v_k|^p\,dx + \int_{B_{\rho'}\setminus B_\rho}(|\nabla w|^p + |w+m_k-\tilde v_k|^p|\nabla\eta|^p)\,dx\right].\end{aligned}\tag{3.26}$$

We split the proof into two cases: either

$$\limsup_k E_k > 0\tag{3.27}$$

or

$$\lim_k E_k = 0. \tag{3.28}$$

Assume (3.27) occurs. Notice that $s \le u_k \le 1$ for every k, then by definition we have that, for every k, $s \le E_k^{1/p}\tilde{v}_k \le 1$ and, since m_k is a median of v_k, $0 \le E_k^{1/p} m_k \le 1$. In particular, we have that

$$|\tilde{v}_k - m_k| \le \frac{2}{E_k^{1/p}},$$

passing to the limit when k goes to infinity we have that

$$\|v\|_\infty \le \liminf_k \frac{2}{E_k^{1/p}} < +\infty \quad \mathcal{L}^n\text{-a.e.}$$

then v belongs to $L^\infty(B_1)$ and there exists a positive constant C independent of k, and a natural number $\overline{k}$ such that

$$|v + m_k - \tilde{v}_k| \le \frac{C}{E_k^{1/p}} \le \frac{C}{s}\tilde{v}_k \quad \mathcal{L}^n\text{-a.e.}$$

for all $k > \overline{k}$. Let $\varphi \in W^{1,p}(B_1)$ with $v - \varphi$ supported on B_ρ, and let $w = \varphi$ in the definition of φ_k, then, for every $k > \overline{k}$, we have

$$|\varphi_k| = |\tilde{v}_k + (v + m_k - \tilde{v}_k)\eta| \le C\tilde{v}_k \tag{3.29}$$

$\mathcal{L}^n$-a.e. on $B_{\rho'} \setminus B_\rho$. From (3.29) we have that

$$\mathcal{I}_k(\varphi_k) \le C\mathcal{I}_k(\tilde{v}_k). \tag{3.30}$$

Notice, in addition, that (3.26) reads as

$$\begin{aligned}\int_{B_{\rho'}} |\nabla\varphi_k|^p\,dx \le& \int_{B_\rho} |\nabla\varphi|^p\,dx \\ &+ C\int_{B_{\rho'}\setminus B_\rho} |\nabla\tilde{v}_k|^p\,dx + C\int_{B_{\rho'}\setminus B_\rho} |\nabla\varphi|^p\,dx + R_k.\end{aligned} \tag{3.31}$$

finally joining (3.23), (3.31), (3.30), and (3.22), we have

$$\alpha_k(\rho') \le \int_{B_\rho} |\nabla\varphi|^p\,dx + C\big[\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho)\big] + C\int_{B_{\rho'}\setminus B_\rho} |\nabla\varphi|^p\,dx + R_k.$$

Letting k go to infinity, we get (3.25).

Suppose now that (3.28) occurs. The functions $|\tilde{v}_k - m_k|^p$, $|v|^p$ are uniformly integrable, namely for every $\varepsilon > 0$ there exists a $\sigma = \sigma_\varepsilon < \varepsilon$ such that if A is a measurable set with $|A| < \sigma$, then

$$\int_A |\tilde{v}_k - m_k|^p\,dx + \int_A |v|^p\,dx < \varepsilon. \tag{3.32}$$

Since $v \in L^p(B_1)$, we can find $M > 1/\varepsilon$ such that

$$|\{\, |v| > M \,\}| < \sigma. \tag{3.33}$$

Setting $w = \varphi_M = \max\{\, -M, \min\{\, \varphi, M \,\}\,\}$, then (3.26) reads as

$$\begin{aligned}\int_{B_{\rho'}} |\nabla\varphi_k|^p\,dx \le& \int_{B_\rho \cap \{\, |\varphi| \le M \,\}} |\nabla\varphi|^p\,dx + C\int_{(B_{\rho'}\setminus B_\rho)\cap\{\, |\varphi|\le M \,\}} |\nabla\varphi|^p\,dx \\ &+ C\left[\int_{B_{\rho'}\setminus B_\rho} |\nabla\tilde{v}_k|^p\,dx + \int_{B_{\rho'}\setminus B_\rho} |\varphi_M + m_k - \tilde{v}_k|^p|\nabla\eta|^p\,dx\right].\end{aligned} \tag{3.34}$$

We can estimate the last integral as follows

$$\int_{B_{\rho'}\setminus B_\rho} |\varphi_M + m_k - \tilde{v}_k|^p |\nabla\eta|^p\,dx \le C\varepsilon + \int_{(B_{\rho'}\setminus B_\rho)\cap\{\,|v|\le M\,\}} |v + m_k - \tilde{v}_k|^p |\nabla\eta|^p\,dx]. \\ = C\varepsilon + R_k, \tag{3.35}$$

where we used (3.33) and (3.32), and C only depends on ρ and ρ'. Furthermore, we have

$$\mathcal{I}_k(\varphi_k) \le R_k + C\mathcal{I}_k(\tilde{v}_k). \tag{3.36}$$

Indeed, as before, $|\tilde{v}_k - m_k| \le C\tilde{v}_k$, while

$$\begin{aligned} E_k^{q/p-1} r_k^{p-1} \int_{J_{\tilde{v}_k}\cap(B_{\rho'}\setminus B_\rho)} |\varphi_M|^q\,d\mathcal{H}^{n-1} &\le M^q E_k^{q/p-1} r_k^{p-1} \mathcal{H}^{n-1}\big(J_{\tilde{v}_k}\cap(B_{\rho'}\setminus B_\rho)\big) \\ &\le M^q E_k^{\frac{q}{p}} \frac{r_k^{p-1}\varepsilon_k}{E_k} \\ &\le \frac{M^q}{s^q} E_k^{\frac{q}{p}}, \end{aligned}$$

which goes to 0 as $k \to \infty$. Finally, joining (3.23), (3.34), (3.35), (3.36), and (3.22), we have

$$\alpha_k(\rho') \le R_k + \int_{B_\rho\cap\{\,|\varphi|\le M\,\}} |\nabla\varphi|^p + C\big[\tilde{\alpha}(\rho') - \tilde{\alpha}(\rho)\big] + C\int_{(B_{\rho'}\setminus B_\rho)\cap\{\,|\varphi|\le M\,\}} |\nabla\varphi|^p\,dx + C\varepsilon.$$

Taking the limit as k tends to infinity, and then the limit as ε tends to 0, we get (3.25).

We are now in a position to prove that v is p-harmonic: taking the limit as ρ' tends to ρ in (3.25), we have that if $\varphi \in W^{1,p}(B_1)$, with $v - \varphi$ supported on B_ρ,

$$\int_{B_\rho} |\nabla v|^p\,dx \le \alpha(\rho) \le \int_{B_\rho} |\nabla\varphi|^p\,dx,$$

for every $\rho \in I$, therefore v is p-harmonic in B_1. Notice that this implies that v is a locally Lipschitz function (see [8, Theorem 7.12]) . Moreover, if $\varphi = v$, we have

$$\int_{B_\rho} |\nabla v|^p\,dx = \alpha(\rho)$$

for every $\rho \in I$, so that α is continuous on the whole interval $[0,1]$, $\alpha(1) = 1$ and $\alpha(\tau) = \lim_k \alpha_k(\tau) \ge \tau^{n-1/2}$. Nevertheless, if τ is sufficiently small this contradicts the fact that v is locally Lipschitz, since

$$\tau^{n-\frac{1}{2}} \le \int_{B_\tau} |\nabla v|^p\,dx \le C\,\tau^n,$$

where C is a positive constant depending only on n and p. □

Proof of Theorem 3.2. Let u be a minimiser for the problem (3.5). By Proposition 3.7 there exist two positive constants $C(\Omega,\beta,p,q), C_1(\Omega,\beta,p,q)$ such that if $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$, then

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \le C(\Omega,\beta,p,q) r^{n-1},$$

and if $x \in K_u$

$$\mathcal{L}^n(B_r(x) \cap \{\,u > 0\,\}) \ge C_1(\Omega,\beta,p,q) r^n.$$

We now prove that there exists a positive constant $c = c(\Omega,\beta,p,q)$ such that

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \ge c(\Omega,\beta,p,q) r^{n-1} \tag{3.37}$$

for every $x \in K_u$ and $B_r(x) \subset \mathbb{R}^n \setminus \Omega$. Assume by contradiction that there exists $x \in J_u$ such that, for $r > 0$ small enough,

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq \varepsilon_0 r^{n-1},$$

where ε_0 is the one in Lemma 3.12. Iterating Lemma 3.12 it can be proven (see [28, Theorem 5.1]) that

$$\lim_{r \to 0^+} r^{1-n} \mathcal{E}_c(u; B_r) = 0,$$

which, in particular, implies

$$\lim_{r \to 0^+} r^{1-n} \left[\int_{B_r(x)} |\nabla u|^p \, dx + \mathcal{H}^{n-1}(J_u \cap B_r(x)) \right] = 0. \tag{3.38}$$

By [44, Theorem 3.6], (3.38) implies that $x \notin J_u$, which is a contradiction. Finally, if $x \in K_u$ and

$$\mathcal{H}^{n-1}(J_u \cap B_{2r}(x)) \leq \varepsilon_0 r^{n-1},$$

there exists $y \in J_u \cap B_r(x)$ such that

$$\mathcal{H}^{n-1}(J_u \cap B_r(y)) \leq \varepsilon_0 r^{n-1}$$

which, again, is a contradiction. Then the assertion is proved. The density estimate (3.37) implies in particular that

$$K_u \setminus \bar{\Omega} \subset \left\{ x \in \mathbb{R}^n \;\middle|\; \limsup_{r \to 0^+} r^{1-n} \left[\int_{B_r(x)} |\nabla u|^p \, dx + \mathcal{H}^{n-1}(J_u \cap B_r(x)) \right] > 0 \right\},$$

hence $\mathcal{H}^{n-1}((K_u \setminus J_u) \setminus \bar{\Omega}) = 0$ (see for instance [44, Lemma 2.6]). $\square$

Remark 3.13. Let u be a minimiser for problem (3.5), then from Theorem 3.5 we have that the function $u^* = (\beta\delta^q)^{-1/p} u$ is an almost-quasi minimiser for the Mumford-Shah functional

$$MS(v) = \int_{\mathbb{R}^n} |\nabla v|^p \, dx + \mathcal{H}^{n-1}(J_v)$$

with the Dirichlet condition $u^* = (\beta\delta^q)^{-1/p}$ on Ω. If Ω is sufficiently smooth, we can apply the results in [22] to have that the density estimate for the jump set of minimisers holds up to the boundary of Ω.

Remark 3.14. Let u be a minimiser to (3.5) and let $A = \{ \overline{u} > 0 \} \setminus K_u$, then the boundary of A is equal to K_u: in first place, assume by contradiction that there exists an $x \in (\partial A) \setminus K_u$, then u is p-harmonic in a small ball centered in x with radius r. Therefore, being

$$\{ u > 0 \} \cap B_r(x) \neq \emptyset,$$

it is necessary that $u > 0$ in the entire ball, and then $x \notin \partial A$, which is a contradiction. In other words,

$$\partial A \subseteq K_u$$

By the same argument we also have that A is open, and moreover $J_u \subseteq \partial A$, then

$$K_u \subseteq \partial A.$$

In particular, the pair (A, u) is a minimiser for the functional

$$\mathcal{E}_c(E, v) = \int_E |\nabla v|^p \, dx + \int_{\partial E} (\underline{v}^q + \overline{v}^q) \, d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n(E \setminus \Omega)$$

over all pairs (E, v) with E open set of finite perimeter containing Ω and $v \in \mathrm{SBV}(\mathbb{R}^n) \cap W^{1,2}(E)$ with $v = 1$ in Ω.

3.2 Thin insulating layer: asymptotic analysis

The content of this section is based on the results of [5]. We now address the problem of qualitatively and quantitatively describing the shape of the optimal configuration of insulating material in the thin layer setting, that is, in the limit in which the thickness and the thermal diffusivity of the insulating material both tend to zero.

More precisely, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded, open set, and let $h\colon \partial\Omega \to \mathbb{R}$ be a positive smooth function. Denoting by ν_0 the exterior unit normal to the boundary of Ω, for every $\varepsilon > 0$ sufficiently small, we define

$$\Sigma_\varepsilon = \{\, \sigma + t\nu_0(\sigma) \mid \sigma \in \partial\Omega,\ 0 < t < \varepsilon h(\sigma) \,\}$$

and we denote by $\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon$. Moreover, we will assume $k = \varepsilon$.

We then consider the minimisation of the energy

$$\mathcal{E}_{c,\varepsilon}(v,h) = \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} v^2\, d\mathcal{H}^{n-1}, \tag{3.39}$$

where $v \in H^1(\Omega_\varepsilon)$, with $v = 1$ in Ω. As we will see in the following chapters, similar problems have been studied before in [16], [42], [6], and more recently in [18] and [36]. The limit can be performed in several ways. In our case, we are going to use Γ-convergence. But to extract as much information as possible about the problem, we are going to perform a first-order expansion in ε.

Instead of penalising the volume of insulator we displace, we will work under a volume constraint assumption. Namely, for a given $m > 0$, the problem of finding the best configuration of insulating material surrounding Ω, in the thin layer setting, can be stated as

$$\min \left\{ \mathcal{E}_{c,\varepsilon}(v,h) \;\middle|\; \begin{array}{l} v \in H^1(\Omega_\varepsilon), \\ v = 1 \text{ in } \Omega, \\ \mathcal{L}^n(\Sigma_\varepsilon) \le \varepsilon m \end{array} \right\}. \tag{3.40}$$

since

$$\mathcal{L}^n(\Sigma_\varepsilon) \sim \varepsilon \int_{\partial\Omega} h\, d\mathcal{H}^{n-1},$$

in the limit, we enforce the volume constraint by restricting our attention to the functions in the space

$$\mathcal{H}_m = \mathcal{H}_m(\partial\Omega) = \left\{ h \in L^1(\partial\Omega) \;\middle|\; \begin{array}{l} \displaystyle\int_{\partial\Omega} h\, d\mathcal{H}^{n-1} \le m \\ h \ge 0 \end{array} \right\}. \tag{3.41}$$

In subsection 3.2.1 we compute the Γ-limit, in the strong $L^2(\mathbb{R}^n)$ topology of the functional $\mathcal{E}_{c,\varepsilon}(\cdot,h)$ and observe that the minimum in the class $\mathcal{H}_m$ is achieved by the constant $h = m/P(\Omega)$, where $P(\Omega)$ denotes the perimeter of Ω.

Displacing the insulator uniformly around the boundary is somehow the trivial solution, the one suggested by common sense, and very common when insulating buildings. However, it is mathematically not satisfactory at all. We expect that portions of the boundary with higher (mean) curvature are less convenient to insulate with respect to those with lower curvature. Such an idea is strongly suggested by the radial cases (see, for example, [37, Proposition 5.1]).

To recover such behaviour, in subsection 3.2.2, we compute the first-order asymptotic development by Γ-convergence (see Definition 2.28) for the functional $\mathcal{E}_{c,\varepsilon}$. Namely, let $\mathcal{E}_{c,0}(h)$ be the minimum of the Γ-limit, let

$$K_0 = \left\{\, v \in L^2(\mathbb{R}^n) \;\middle|\; v = 1 \text{ in } \Omega \,\right\},$$

and denote by H the mean curvature of Ω (see Definition 2.23). We prove the following

Theorem 3.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with C^3 boundary, and fix a C^2 function $h\colon \partial\Omega \to (0,+\infty)$. Then the functional*

$$\delta\mathcal{E}_{c,\varepsilon}(\cdot,h) = \frac{\mathcal{E}_{c,\varepsilon}(\cdot,h) - \mathcal{E}_{c,0}(h)}{\varepsilon}$$

Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, as $\varepsilon \to 0^+$, to

$$\mathcal{E}_c^{(1)}(v,h) = \begin{cases} \beta \displaystyle\int_{\partial\Omega} \frac{Hh(2+\beta h)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1} & \text{if } v \in K_0, \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n) \setminus K_0. \end{cases} \tag{3.42}$$

Finally, in subsection 3.2.3 study with the minimum problem

$$\inf\left\{ \mathcal{E}_{c,0}(h) + \varepsilon\mathcal{E}_c^{(1)}(h) \;\middle|\; h \in \mathcal{H}_m \right\},$$

where $\mathcal{E}_c^{(1)}(h) = \mathcal{E}_c^{(1)}(\chi_\Omega, h)$. From the properties of Γ-convergence, the above problem is a first-order approximation, in $\varepsilon > 0$, of the problem (3.40). Indeed we have that (see Remark 2.29)

$$\mathcal{E}_{c,\varepsilon}(u_\varepsilon,h) = \mathcal{E}_{c,0}(h) + \varepsilon\mathcal{E}_c^{(1)}(h) + R(\Omega,h,\varepsilon),$$

where u_ε is the minimiser to (3.39), and

$$\lim_{\varepsilon\to 0^+} \frac{R(\Omega,h,\varepsilon)}{\varepsilon} = 0.$$

In particular, we will prove that, as the intuition suggests, if ε is small enough, then the optimal configuration for the insulating layer concentrates close to the points of $\partial\Omega$ where the mean curvature is relatively small.

3.2.1 The limit problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive Lipschitz function $h\colon \partial\Omega \to \mathbb{R}$. In the notations of Section 2.2 let

$$\Gamma_t = \{\, x \in \mathbb{R}^n \mid d(x) < t \,\} \setminus \Omega,$$

$$\gamma_t = \partial(\Omega \cup \Gamma_t) = \{\, x \in \mathbb{R}^n \mid d(x) = t \,\} \setminus \Omega.$$

Our assumptions on $\partial\Omega$ ensure that there exists $d_0 > 0$ such that for every $x \in \Gamma_{d_0}$, we can uniquely write

$$x = \sigma(x) + d(x)\nu_0(\sigma(x)),$$

where $\sigma(x)$ is the metric projection of x on Ω and $d(x)$ is its distance from Ω. Moreover, on Γ_{d_0}, extending ν_0 as $\nu_0(x) = \nu_0(\sigma(x))$ we have that ν_0 is orthogonal to the level set of the distance. In the following we will assume h to be extended to Γ_{d_0} as $h(x) = h(\sigma(x))$ and $\varepsilon > 0$ such that $\varepsilon\|h\|_\infty < d_0$, so that $\Sigma_\varepsilon \subset \Gamma_{d_0}$.

Remark 3.16. Using the coarea formula (Theorem 2.25) with the distance function d, we have that for every $g \in L^1(\Omega_\varepsilon)$

$$\int_{\Sigma_\varepsilon} g(x)\, dx = \int_0^{+\infty} \int_{\gamma_t} g(\xi)\, \chi_{\Sigma_\varepsilon}(\xi)\, d\mathcal{H}^{n-1}(\xi)\, dt.$$

Let

$$\phi_t \colon x \in \Gamma_{d_0} \mapsto x + t\nu_0(x) \in \Gamma_{td_0},$$

then $\gamma_t = \phi_t(\partial\Omega)$, and by the area formula on surfaces (Theorem 2.21)

$$\begin{aligned}\int_{\Sigma_\varepsilon} g(x)\,dx &= \int_0^{+\infty}\int_{\partial\Omega} g(\sigma+t\nu_0)\chi_{\Sigma_\varepsilon}(\sigma+t\nu_0)J^\tau\phi_t(\sigma)\,d\mathcal{H}^{n-1}(\sigma)\,dt\\ &= \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} g(\sigma+t\nu_0)\prod_{i=1}^{n-1}(1+tk_i(\sigma))\,dt\,d\mathcal{H}^{n-1}(\sigma).\end{aligned}$$

Similarly,

$$\int_{\partial\Omega_\varepsilon} g(\xi)\,d\mathcal{H}^{n-1}(\xi) = \int_{\partial\Omega} g(\sigma+\varepsilon h\nu_0)\prod_{i=1}^{n-1}(1+\varepsilon h(\sigma)k_i(\sigma))\sqrt{1+\varepsilon^2|\nabla h|^2}\,d\mathcal{H}^{n-1}(\sigma).$$

so that we have

$$\int_{\Sigma_\varepsilon} g(x)\,dx = \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} g(\sigma+t\nu_0)\big(1+tH(\sigma)+\varepsilon^2R_1(\sigma,t,\varepsilon)\big)\,dt\,d\mathcal{H}^{n-1} \tag{3.43}$$

and

$$\int_{\partial\Omega_\varepsilon} g(\sigma)\,d\mathcal{H}^{n-1} = \int_{\partial\Omega} g(\sigma+\varepsilon h\nu_0)\big(1+\varepsilon h(\sigma)H(\sigma)+\varepsilon^2R_2(\sigma,\varepsilon)\big)\,d\mathcal{H}^{n-1}, \tag{3.44}$$

where the remainder terms R_1 and R_2 are bounded functions. In other words, there exists $Q = Q(\Omega,h) > 0$ such that $|R_1|, |R_2| \le Q$.

In particular, notice that there exists a positive constant $C = C(\Omega, \|h\|_{0,1})$ such that for every $0 < \varepsilon < 1$

$$\frac{1}{C}\int_{\Sigma_\varepsilon} g\,dx \le \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} g(\sigma+t\nu_0)\,dt\,d\mathcal{H}^{n-1} \le C\int_{\Sigma_\varepsilon} g\,dx, \tag{3.45}$$

and

$$\frac{1}{C}\int_{\partial\Omega_\varepsilon} g\,dx \le \int_{\partial\Omega} g(\sigma+\varepsilon h\nu_0)\,d\mathcal{H}^{n-1} \le C\int_{\partial\Omega_\varepsilon} g\,dx. \tag{3.46}$$

Let

$$K_\varepsilon = \left\{\, v \in H^1(\Omega_\varepsilon) \;\middle|\; v = 1 \text{ in } \Omega \,\right\}, \tag{3.47}$$

and

$$K_0 = \left\{\, v \in L^2(\mathbb{R}^n) \;\middle|\; v = 1 \text{ in } \Omega \,\right\}, \tag{3.48}$$

and consider the functional

$$\mathcal{E}_{c,\varepsilon}(v,h) = \begin{cases} \varepsilon\displaystyle\int_{\Sigma_\varepsilon}|\nabla v|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1} & \text{if } v \in K_\varepsilon,\\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus K_\varepsilon.\end{cases}$$

denoting by

$$\mathcal{E}_{c,0}(v,h) = \begin{cases} \beta\displaystyle\int_{\partial\Omega}\frac{1}{1+\beta h}\,d\mathcal{H}^{n-1} & \text{if } v \in K_0,\\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus K_0,\end{cases}$$

following the approach of [36], we have the following

Proposition 3.17. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a Lipchitz function $h\colon \partial\Omega \to (0,+\infty)$. Then $\mathcal{E}_{c,\varepsilon}(\cdot,h)$ Γ-converges, as $\varepsilon \to 0^+$, in the strong $L^2(\mathbb{R}^n)$ topology, to $\mathcal{E}_{c,0}(\cdot,h)$.*

Proof. We start by proving the Γ-liminf inequality: Let $v \in L^2(\mathbb{R}^n)$ and let $v_\varepsilon \in L^2(\mathbb{R}^n)$ such that v_ε converges to v in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Up to passing to a sub-sequence, we can assume that

$$\liminf_{\varepsilon \to 0^+} \mathcal{E}_{c,\varepsilon}(v_\varepsilon, h) = \lim_{\varepsilon \to 0^+} \mathcal{E}_{c,\varepsilon}(v_\varepsilon, h),$$

moreover, we can assume that such a limit is finite and that $v_\varepsilon \in K_\varepsilon$. Therefore we have that $v \in K_0$ and, by (3.43), (3.44) we have that

$$\int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \, dx \geq \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu_0)|^2 (1 - \varepsilon Q_0) \, dt \, d\mathcal{H}^{n-1} \tag{3.49}$$

and

$$\int_{\partial\Omega_\varepsilon} v_\varepsilon^2 \, d\mathcal{H}^{n-1} \geq \int_{\partial\Omega} v_\varepsilon^2(\sigma + \varepsilon h(\sigma)\nu_0)(1 - \varepsilon Q_0) \, d\mathcal{H}^{n-1} \tag{3.50}$$

for some constant $Q_0 > 0$. On the other hand, we have that, for $\mathcal{H}^{n-1}$-almost every $\sigma \in \partial\Omega$,

$$\begin{aligned} \int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu_0)|^2 \, dt \geq & \frac{1}{\varepsilon h(\sigma)} \left(\int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu_0)| \, dt \right)^2 \\ \geq & \frac{(v_\varepsilon(\sigma + \varepsilon h \nu_0) - 1)^2}{\varepsilon h(\sigma)}, \end{aligned}$$

then by Young's inequality, we have that, for every $\lambda > 0$ and for $\mathcal{H}^{n-1}$-almost every $\sigma \in \partial\Omega$,

$$\int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu_0)|^2 \, dt \geq \frac{(1-\lambda) v_\varepsilon(\sigma + \varepsilon h \nu_0)^2}{\varepsilon h(\sigma)} + \frac{1}{\varepsilon h(\sigma)} \left(1 - \frac{1}{\lambda} \right). \tag{3.51}$$

Putting together (3.49), (3.50) and (3.51) we finally have

$$\mathcal{E}_{c,\varepsilon}(v_\varepsilon, h) \geq \int_{\partial\Omega} \left(\left(\frac{1-\lambda}{h} + \beta \right) v_\varepsilon(\sigma + \varepsilon h \nu_0)^2 + \frac{1}{h} \left(1 - \frac{1}{\lambda} \right) \right) d\mathcal{H}^{n-1} - \varepsilon Q_0 R_\varepsilon(\varepsilon, v_\varepsilon),$$

where, if ε is sufficiently small, using again (3.43) and (3.44), we have

$$R_\varepsilon(\varepsilon, v_\varepsilon) \leq 2\mathcal{E}_{c,\varepsilon}(v_\varepsilon, h).$$

Finally, letting $\lambda = \lambda(\sigma) = 1 + \beta h(\sigma)$, and passing to the limit as $\varepsilon \to 0^+$ we have that

$$\liminf_{\varepsilon \to 0^+} \mathcal{E}_{c,\varepsilon}(v_\varepsilon, h) \geq \beta \int_{\partial\Omega} \frac{1}{1 + \beta h} \, d\mathcal{H}^{n-1} = \mathcal{E}_{c,0}(v, h)$$

and the Γ-liminf inequality is proved.

Γ-limsup inequality: Let $v \in L^2(\mathbb{R}^n)$, if $v \notin K_0$ the Γ-limsup inequality is trivial, therefore let $v \in K_0$. Let

$$v_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 1 - \dfrac{\beta d(x)}{\varepsilon(1 + \beta h(x))} & \text{if } x \in \Sigma_\varepsilon, \\ v(x) & \text{if } x \notin \Omega_\varepsilon, \end{cases}$$

where we recall that, if $x = \sigma + t\nu_0(\sigma)$, then $h(x) = h(\sigma)$. Trivially v_ε converges to v in $L^2(\mathbb{R}^n)$ and $v_\varepsilon \in K_\varepsilon$. For every $x \in \Sigma_\varepsilon$,

$$\nabla v_\varepsilon(x) = -\frac{\beta \nabla d(x)}{\varepsilon(1 + \beta h(x))} + \frac{\beta^2 d(x) \nabla h(x)}{\varepsilon(1 + \beta h(x))^2}.$$

Recalling that $0 \le d \le \varepsilon h$, $\nabla d = \nu_0$ and $\nabla h \cdot \nu_0 = 0$, we have

$$|\nabla v_\varepsilon|^2 = \frac{\beta^2}{\varepsilon^2(1+\beta h)^2} + \frac{\beta^4 d^2 |\nabla h|^2}{\varepsilon^2(1+\beta h)^4} \le \frac{\beta^2}{\varepsilon^2(1+\beta h)^2} + \frac{\beta^4 h^2 |\nabla h|^2}{(1+\beta h)^4},$$

where the second term is bounded since h is Lipschitz. Hence, substituting $\varepsilon h\tau = t$ in (3.43), we get

$$\begin{aligned}
\varepsilon \int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \, dx &\le \frac{\beta^2}{\varepsilon} \int_{\Sigma_\varepsilon} \frac{1}{(1+\beta h)^2} \, dx + \varepsilon C |\Sigma_\varepsilon| \\
&\le \int_{\partial\Omega} \int_0^1 \frac{\beta^2 h}{(1+\beta h)^2} (1+\varepsilon Q_0) \, d\tau \, d\mathcal{H}^{n-1} + \varepsilon C |\Sigma_\varepsilon| \\
&= \int_{\partial\Omega} \frac{\beta^2 h}{(1+\beta h)^2} \, d\mathcal{H}^{n-1} + o(\varepsilon).
\end{aligned}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$v_\varepsilon(\sigma + \varepsilon h(\sigma)\nu_0(\sigma)) = \frac{1}{1+\beta h(\sigma)},$$

from which we get

$$\beta \int_{\partial\Omega_\varepsilon} v_\varepsilon^2 \, d\mathcal{H}^{n-1} \le \int_{\partial\Omega} \frac{\beta}{(1+\beta h)^2} (1+\varepsilon Q_0) \, d\mathcal{H}^{n-1} = \int_{\partial\Omega} \frac{\beta}{(1+\beta h)^2} \, d\mathcal{H}^{n-1} + o(\varepsilon).$$

Hence we have

$$\mathcal{E}_{c,\varepsilon}(v_\varepsilon, h) \le \beta \int_{\partial\Omega} \frac{1}{1+\beta h} \, d\mathcal{H}^{n-1} + o(\varepsilon),$$

so that

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}_{c,\varepsilon}(v_\varepsilon, h) \le \beta \int_{\partial\Omega} \frac{1}{1+\beta h} \, d\mathcal{H}^{n-1}$$

and the Γ-limsup inequality is proved. □

In the following, for simplicity, we will denote by

$$\mathcal{E}_{c,0}(h) = \beta \int_{\partial\Omega} \frac{1}{1+\beta h} \, d\mathcal{H}^{n-1}.$$

The minimum in the class of functions h with a given mass, of such a functional is achieved when h is constant. Indeed, we have the following proposition.

Proposition 3.18. *Let Ω be a bounded open set with Lipschitz boundary, let $P(\Omega) = P$, and let $m > 0$. Then the problem*

$$\min \{ \, \mathcal{E}_{c,0}(h) \mid h \in \mathcal{H}_m \, \} \tag{3.52}$$

admits

$$h_0 = \frac{m}{P}$$

as the unique solution.

Proof. Let $h \in \mathcal{H}_m$. By Hölder's inequality, we have that

$$\begin{aligned}
P = \int_{\partial\Omega} d\mathcal{H}^{n-1} &\le \left(\int_{\partial\Omega} \frac{1}{1+\beta h} \, d\mathcal{H}^{n-1} \right)^{1/2} \left(\int_{\partial\Omega} (1+\beta h) \, d\mathcal{H}^{n-1} \right)^{1/2} \\
&\le \left(\int_{\partial\Omega} \frac{1}{1+\beta h} \, d\mathcal{H}^{n-1} \right)^{1/2} (P + \beta m)^{1/2},
\end{aligned}$$

so that

$$\mathcal{E}_{c,0}(h) \geq \frac{\beta P^2}{P+\beta m} = \mathcal{E}_{c,0}(h_0).$$

Finally, the uniqueness of the solution is given by the strict convexity of the function

$$x \longmapsto \frac{1}{1+\beta x}$$

for $x \geq 0$. □

3.2.2 first-order development

Let Ω be a bounded, open set with C^3 boundary, and fix a positive C^2 function $h\colon \partial\Omega \to \mathbb{R}$. In this section, We prove a first-order asymptotic development by Γ-convergence (see Definition 2.28) for the energy $\mathcal{E}_{c,\varepsilon}(\cdot, h)$, namely, we study the Γ-convergence of the family of functionals

$$\delta\mathcal{E}_{c,\varepsilon}(v) = \frac{\mathcal{E}_{c,\varepsilon}(\cdot, h) - \mathcal{E}_{c,0}(h)}{\varepsilon}, \tag{3.53}$$

and we prove Theorem 3.15. In the following we consider the functions $h, H\colon \partial\Omega \to \mathbb{R}$ extended on the set Σ_ε as $h(\sigma + t\nu_0) = h(\sigma)$ and $H(\sigma + t\nu_0) = H(\sigma)$.

For every $\varepsilon > 0$ let $u_\varepsilon \in K_\varepsilon$ be the minimiser to $\mathcal{E}_{c,\varepsilon}$, where K_ε is defined in (3.47). By the assumptions on Ω and h, we have that u_ε is a $C^2(\Sigma_\varepsilon)$ function and it is a solution to

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ u_\varepsilon = 1 & \text{on } \partial\Omega, \\ \varepsilon\partial_{\nu_\varepsilon} u_\varepsilon + \beta u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{3.54}$$

where ν_ε is the outer unit normal to the boundary of Ω_ε.

Let $\alpha \in (0, \alpha_0)$, where

$$\alpha_0 = 1 - \max_{\sigma\in\partial\Omega} \frac{\beta h(\sigma)}{1+\beta h(\sigma)},$$

and, for every $x \in \Sigma_\varepsilon$ let

$$w_{\varepsilon,\alpha}(x) = \begin{cases} 1 - \left(\dfrac{d(x)}{\varepsilon h(x)}\right)^{1-\alpha} \dfrac{\beta h(x)}{(1-\alpha)(1+\beta h(x))} & \text{if } H(x) \geq 0, \\[2ex] 1 - \left(\dfrac{d(x)}{\varepsilon h(x)}\right)^{1+\alpha} \dfrac{\beta h(x)}{(1+\alpha)(1+\beta h(x))} & \text{if } H(x) < 0. \end{cases}$$

We have that $w_{\varepsilon,\alpha} > 0$ and the following proposition.

Proposition 3.19. *For every $\alpha \in (0, \alpha_0)$ there exists $\varepsilon_\alpha > 0$ such that if $0 < \varepsilon < \varepsilon_\alpha$, then*

$$H(x) u_\varepsilon(x) \geq H(x) w_{\varepsilon,\alpha}(x) \qquad \textit{for } x \in \Sigma_\varepsilon.$$

Proof. Fix $\alpha \in (0, \alpha_0)$. For simplicity, we denote by

$$v_\gamma := 1 - \left(\frac{d(x)}{\varepsilon h(x)}\right)^\gamma \frac{\beta h(x)}{\gamma(1+\beta h(x))},$$

and we aim to show that there exists an $\varepsilon_\alpha > 0$ such that for any $0 < \varepsilon < \varepsilon_\alpha$, we have that $v_{1-\alpha}$ is a subsolution to (3.54), while $v_{1+\alpha}$ is a supersolution to the same problem. Namely,

$$\begin{cases} -\Delta v_{1-\alpha} \leq 0 & \text{in } \Sigma_\varepsilon, \\ v_{1-\alpha} = 1 & \text{on } \partial\Omega, \\ \varepsilon \dfrac{\partial v_{1-\alpha}}{\partial \nu_\varepsilon} + \beta v_{1-\alpha} \leq 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \qquad \begin{cases} -\Delta v_{1+\alpha} \geq 0 & \text{in } \Sigma_\varepsilon, \\ v_{1+\alpha} = 1 & \text{on } \partial\Omega, \\ \varepsilon\partial_{\nu_\varepsilon} v_{1+\alpha} + \beta v_{1+\alpha} \geq 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3.55}$$

In the following, we will always assume that $\varepsilon < 1$. Let us recall that

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{d(x)}{h(x)} \le \varepsilon \right\}, \qquad \partial\Omega_\varepsilon = \{ x \in \mathbb{R}^n \mid d(x) - \varepsilon h(x) = 0 \}.$$

By standard computations, we get

$$\nabla\left(\frac{d}{h}\right) = \frac{\nabla d}{h} - \frac{d\nabla h}{h^2}, \qquad \left|\nabla\left(\frac{d}{h}\right)\right| = \frac{1}{h}\sqrt{1 + \left(\frac{d}{h}\right)^2 |\nabla h|^2}.$$

Then, recalling that $\nabla d = \nu_0$ and that $\nabla h \cdot \nu_0 = 0$, the normal ν_ε to the set Ω_ε is given by

$$\nu_\varepsilon = \frac{1}{\sqrt{1+\varepsilon^2|\nabla h|^2}}(\nu_0 - \varepsilon\nabla h).$$

By direct computations, for any $\gamma \in (0,2) \setminus \{1\}$ we have

$$\Delta v_\gamma = -\frac{\beta h}{\gamma\varepsilon^\gamma(1+\beta h)}\Delta\left[\left(\frac{d}{h}\right)^\gamma\right] - \frac{2}{\gamma\varepsilon^\gamma}\nabla\left[\left(\frac{d}{h}\right)^\gamma\right] \cdot \nabla\left[\frac{\beta h}{1+\beta h}\right] - \frac{1}{\gamma\varepsilon^\gamma}\left(\frac{d}{h}\right)^\gamma \Delta\left[\frac{\beta h}{1+\beta h}\right]. \tag{3.56}$$

We then compute

$$\nabla\left[\left(\frac{d}{h}\right)^\gamma\right] = \gamma\left(\frac{d}{h}\right)^{\gamma-1}\left(\frac{\nu_0}{h} - \frac{d\nabla h}{h^2}\right), \qquad \nabla\left[\frac{\beta h}{1+\beta h}\right] = \frac{\beta\nabla h}{(1+\beta h)^2}, \tag{3.57}$$

from which we get

$$\nabla\left[\left(\frac{d}{h}\right)^\gamma\right] \cdot \nabla\left[\frac{\beta h}{1+\beta h}\right] = -\gamma\left(\frac{d}{h}\right)^\gamma \frac{\beta|\nabla h|^2}{h(1+\beta h)^2}. \tag{3.58}$$

In addition, we have

$$\begin{aligned}\Delta\left[\left(\frac{d}{h}\right)^\gamma\right] &= \gamma(\gamma-1)\left(\frac{d}{h}\right)^{\gamma-2}\left|\nabla\left(\frac{d}{h}\right)\right|^2 + \gamma\left(\frac{d}{h}\right)^{\gamma-1}\left(\frac{\Delta d}{h} - \frac{d\Delta h}{h^2} + 2\frac{d|\nabla h|^2}{h^3}\right)\\ &= \gamma\left(\frac{d}{h}\right)^{\gamma-2}\left(-\frac{1-\gamma}{h^2} - \left(\frac{d}{h}\right)^2\frac{(1-\gamma)|\nabla h|^2}{h^2} + \frac{d}{h}\frac{\Delta d}{h} - \left(\frac{d}{h}\right)^2\frac{\Delta h}{h} + \left(\frac{d}{h}\right)^2\frac{2|\nabla h|^2}{h^2}\right),\end{aligned} \tag{3.59}$$

so that, by (3.56), (3.59), and (3.58), we get

$$\left(\frac{d(x)}{h(x)}\right)^{2-\gamma}\varepsilon^\gamma\Delta v_\gamma(x) = \frac{\beta(1-\gamma)}{h(x)(1+\beta h(x))} + R_1(x,\varepsilon,\gamma), \tag{3.60}$$

where $R_1(x,\varepsilon,\gamma)$ is a suitable remainder term. Since $d \le \varepsilon h$,

$$0 < \inf_{\Sigma_\varepsilon} h \le \sup_{\Sigma_\varepsilon} h < +\infty,$$

and $|\nabla h|$, Δh, Δd are bounded, then there exist $C_\gamma, \varepsilon_0 > 0$ such that

$$|R_1(x,\varepsilon,\gamma)| \le C_\gamma\varepsilon \tag{3.61}$$

for any $\varepsilon < \varepsilon_0$. Thus, using (3.61) in (3.60) we have that there exists $\varepsilon_\alpha > 0$ such that if $0 < \varepsilon < \varepsilon_\alpha$, then

$$-\Delta v_{1-\alpha} < 0, \qquad -\Delta v_{1+\alpha} > 0. \tag{3.62}$$

On the other hand, for every $x \in \partial\Omega_\varepsilon$, since $d(x) = \varepsilon h(x)$ and (3.57) hold true, we get

$$\nabla v_\gamma(x) = -\frac{\beta}{\varepsilon(1+\beta h(x))}(\nu_0(x) - \varepsilon\nabla h(x)) - \frac{\beta\nabla h(x)}{\gamma(1+\beta h(x))^2},$$

which yields

$$\begin{aligned}\partial_{\nu_\varepsilon} v_\gamma(x) &= -\frac{\beta\sqrt{1+\varepsilon^2|\nabla h|^2}}{\varepsilon(1+\beta h)} - \frac{\beta}{\gamma}\nu_\varepsilon \cdot \frac{\nabla h}{(1+\beta h)^2} \\ &= -\frac{\beta\sqrt{1+\varepsilon^2|\nabla h|^2}}{\varepsilon(1+\beta h)} + \frac{\beta\varepsilon|\nabla h|^2}{\gamma(1+\beta h)^2\sqrt{1+\varepsilon^2|\nabla h|^2}},\end{aligned} \tag{3.63}$$

while

$$v_\gamma(x) = 1 - \frac{\beta h(x)}{\gamma(1+\beta h(x))}. \tag{3.64}$$

Hence, we get by (3.63) and (3.64)

$$\varepsilon\partial_{\nu_\varepsilon} v_\gamma + \beta v_\gamma = -(1-\gamma)\frac{\beta^2 h}{\gamma(1+\beta h)} + R_2(\sigma,\varepsilon,\gamma) \quad \text{on } \partial\Omega_\varepsilon,$$

where, as before, up to choosing a smaller ε_0,

$$|R_2(\sigma,\varepsilon,\gamma)| \le C_\gamma\varepsilon.$$

Again, for small enough ε, on $\partial\Omega_\varepsilon$ we get

$$\varepsilon\partial_{\nu_\varepsilon} v_{1-\alpha} + \beta v_{1-\alpha} < 0, \qquad \varepsilon\partial_{\nu_\varepsilon} v_{1+\alpha} + \beta v_{1+\alpha} > 0. \tag{3.65}$$

Finally, joining (3.62) and (3.65), by standard comparison results for elliptic operators, the proposition is proved. □

Remark 3.20. The previous proposition implies that for every $\sigma \in \partial\Omega$ and $t \in (0, h(\sigma)$

$$u_\varepsilon(\sigma + \varepsilon t\nu_0) \to 1 - \frac{\beta t}{1+\beta h(\sigma)}.$$

Indeed, for every γ

$$v_\gamma(\sigma + \varepsilon t\nu_0) = 1 - \left(\frac{t}{h(\sigma)}\right)^\gamma \frac{\beta h(\sigma)}{\gamma(1+\beta h(\sigma))},$$

hence, for every $\alpha \in (0,\alpha_0)$ and $\varepsilon < \varepsilon_\alpha$, we have $v_{1-\alpha}(\sigma+\varepsilon t\nu_0) \le u_\varepsilon(\sigma+\varepsilon t\nu_0) \le v_{1+\alpha}(\sigma+\varepsilon t\nu_0)$. passing to the limit for ε to zero we have that for every α small,

$$1 - \left(\frac{t}{h(\sigma)}\right)^{1-\alpha}\frac{\beta h(\sigma)}{(1-\alpha)(1+\beta h(\sigma))} \le \lim_{\varepsilon\to 0^+} u_\varepsilon(\sigma+\varepsilon t\nu_0) \le 1 - \left(\frac{t}{h(\sigma)}\right)^{1+\alpha}\frac{\beta h(\sigma)}{(1+\alpha)(1+\beta h(\sigma))}.$$

Then passing to the limit for α to zero, we have the claim.

We can now prove Theorem 3.15.

Proof of Theorem 3.15. We start by proving the Γ-liminf inequality: Without loss of generality, we can prove the inequality for the sequence of minimisers u_ε. Here we recall the definitions of $\mathcal{E}_{c,\varepsilon}$ and $\mathcal{E}_{c,0}$, omitting the dependence on h.

$$\mathcal{E}_{c,\varepsilon}(u_\varepsilon) = \varepsilon\int_{\Sigma_\varepsilon}|\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1}, \tag{3.66}$$

$$\mathcal{E}_{c,0} = \beta \int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1}. \tag{3.67}$$

By (3.43) and (3.44) we have

$$\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\, dx \geq \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu_0)|^2 \big(1 + tH(\sigma) - \varepsilon^2 Q\big)\, dt\, d\mathcal{H}^{n-1} \tag{3.68}$$

and

$$\frac{\beta}{\varepsilon} \int_{\partial\Omega_\varepsilon} u_\varepsilon^2\, \mathcal{H}^{n-1} \geq \frac{\beta}{\varepsilon} \int_{\partial\Omega} u_\varepsilon^2(\sigma + \varepsilon h(\sigma)\nu_0(\sigma)) \big(1 + \varepsilon h(\sigma) H(\sigma) - \varepsilon^2 Q\big)\, d\mathcal{H}^{n-1} \tag{3.69}$$

for some constant $Q > 0$. For ε sufficiently small, for every $\sigma \in \partial\Omega$, and $0 < t < \varepsilon h(\sigma)$, we have that $1 + tH(\sigma) > 0$, so that, using Hölder's inequality and integrating by parts,

$$\begin{aligned}
\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu_0)|^2 (1 + tH(\sigma))\, dt &\geq \frac{1}{\varepsilon h} \left(\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu_0)| \sqrt{1+tH}\, dt \right)^2 \\
&\geq \frac{1}{\varepsilon h} \left(\int_0^{\varepsilon h(\sigma)} \frac{d}{dt}(u_\varepsilon(\sigma + t\nu_0)) \sqrt{1+tH}\, dt \right)^2 \\
&\geq \frac{1}{\varepsilon h} \left(u_\varepsilon(\sigma + \varepsilon h\nu_0)\sqrt{1+\varepsilon hH} - \left(1 + \int_0^{\varepsilon h} \frac{H u_\varepsilon(\sigma + t\nu_0)}{2\sqrt{1+tH}}\, dt \right) \right)^2.
\end{aligned}$$

Up to choosing a smaller ε, we can apply Young's inequality, having that for every $\lambda > 0$

$$\begin{aligned}
\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu_0)|^2 (1 + tH(\sigma))\, dt \geq &\frac{(1-\lambda)(1+\varepsilon hH) u_\varepsilon(\sigma + \varepsilon h\nu_0)^2}{\varepsilon h} \\
&+ \frac{1}{\varepsilon h}\left(1 - \frac{1}{\lambda}\right) \left(1 + \int_0^{\varepsilon h(\sigma)} \frac{H u_\varepsilon(\sigma + t\nu_0)}{2\sqrt{1+tH}}\, dt \right)^2.
\end{aligned} \tag{3.70}$$

We then have, joining (3.66), (3.68), (3.70), (3.69), and (3.67),

$$\begin{aligned}
\delta\mathcal{E}_{c,\varepsilon}(u_\varepsilon) = \frac{\mathcal{E}_{c,\varepsilon}(u_\varepsilon) - \mathcal{E}_{c,0}}{\varepsilon} \geq &\int_{\partial\Omega} \frac{1}{\varepsilon h(\sigma)} ((1-\lambda)(1+\varepsilon hH) + \beta h(1+\varepsilon hH)) u_\varepsilon^2(\sigma + \varepsilon h\nu_0)\, d\mathcal{H}^{n-1} \\
&+ \int_{\partial\Omega} \frac{1}{\varepsilon h} \left(\left(1 - \frac{1}{\lambda}\right) \left(1 + \int_0^{\varepsilon h} \frac{H u_\varepsilon(\sigma + t\nu_0)}{2\sqrt{1+tH}}\, dt \right)^2 - \frac{\beta h}{1+\beta h} \right) d\mathcal{H}^{n-1} \\
&- Q\varepsilon R(\varepsilon, u_\varepsilon)
\end{aligned} \tag{3.71}$$

where, if ε is small enough,

$$\begin{aligned}
R(\varepsilon, u_\varepsilon) &= \varepsilon \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu_0)|^2\, d\mathcal{H}^{n-1} + \beta \int_{\partial\Omega} u_\varepsilon(\sigma + \varepsilon h(\sigma)\nu_0(\sigma))^2\, d\mathcal{H}^{n-1} \\
&\leq 2\mathcal{E}_{c,\varepsilon}(u_\varepsilon).
\end{aligned}$$

Letting $\lambda = \lambda(\sigma) = 1 + \beta h(\sigma)$ in (3.71), and using the inequality $(1+x)^2 \geq 1 + 2x$,

$$\delta\mathcal{E}_{c,\varepsilon}(u_\varepsilon) \geq \int_{\partial\Omega} \frac{\beta hH}{\varepsilon(1+\beta h)} \int_0^{\varepsilon} \frac{u_\varepsilon(\sigma + th\nu_0)}{\sqrt{1+thH}}\, dt\, d\mathcal{H}^{n-1} + O(\varepsilon).$$

Moreover, for every $t \in (0, \varepsilon)$ we have that $(1 + thH)^{-1/2} = 1 + O(\varepsilon)$, so that

$$\delta\mathcal{E}_{c,\varepsilon}(u_\varepsilon) \geq \beta \int_{\partial\Omega} \frac{hH}{(1+\beta h)} \fint_0^\varepsilon u_\varepsilon(\sigma + th\nu_0)\, dt\, d\mathcal{H}^{n-1} + O(\varepsilon). \tag{3.72}$$

Finally, let $\alpha \in (0, 1)$ and let

$$\gamma = \gamma(\sigma) = \begin{cases} 1 - \alpha & \text{if } H(\sigma) \geq 0 \\ 1 + \alpha & \text{if } H(\sigma) < 0. \end{cases}$$

Let us recall that

$$w_{\varepsilon,\alpha}(\sigma + th(\sigma)\nu_0(\sigma)) = 1 - t^\gamma \frac{\beta h(\sigma)}{\varepsilon^\gamma \gamma(1 + \beta h(\sigma))}.$$

By Proposition 3.19 we have that for every $0 < \varepsilon < \varepsilon_\alpha$

$$\begin{aligned} \delta\mathcal{E}_{c,\varepsilon}(u_\varepsilon) &\geq \beta \int_{\partial\Omega} \frac{hH}{(1+\beta h)} \fint_0^\varepsilon w_{\varepsilon,\alpha}(\sigma + th\nu_0)\, dt\, d\mathcal{H}^{n-1} + O(\varepsilon) \\ &= \int_{\partial\Omega} \left(1 - \frac{\beta h}{(1+\beta h)\gamma(\gamma+1)}\right) \frac{\beta hH}{1+\beta h}\, d\mathcal{H}^{n-1} + O(\varepsilon), \end{aligned}$$

so that

$$\liminf_{\varepsilon\to 0^+} \delta\mathcal{E}_{c,\varepsilon}(u_\varepsilon) \geq \int_{\partial\Omega} \left(1 - \frac{\beta h}{(1+\beta h)\gamma(\gamma+1)}\right) \frac{\beta hH}{1+\beta h}\, d\mathcal{H}^{n-1}.$$

Letting α go to 0, we have that γ tends to 1, and

$$\liminf_{\varepsilon\to 0^+} \delta\mathcal{E}_{c,\varepsilon}(u_\varepsilon) \geq \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1},$$

and the Γ-Liminf is proved.

We now prove the Γ-limsup inequality:
Let

$$\varphi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 1 - \dfrac{\beta d(x)}{\varepsilon(1+\beta h(x))} & \text{if } x \in \Sigma_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases}$$

where we recall that if $x = \sigma + t\nu_0(\sigma)$, then $h(x) = h(\sigma)$.

We have that $\varphi_\varepsilon \in H^1(\Omega)$ and φ_ε converges in $L^2(\mathbb{R}^n)$, to the characteristic function of Ω. Computing the gradient of φ_ε, for any $x \in \Sigma_\varepsilon$,

$$|\nabla\varphi_\varepsilon|^2 \leq \frac{\beta^2}{\varepsilon^2(1+\beta h)^2} + C,$$

where we used again the boundedness of h, and the fact that $d \leq \varepsilon h$. Hence, substituting $\varepsilon h\tau = t$ in (3.43), and noticing that $d(\sigma + \varepsilon\tau h(\sigma)\nu_0(\sigma)) = \varepsilon h\tau$, we get

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} |\nabla\varphi_\varepsilon|^2\, dx &\leq \frac{\beta^2}{\varepsilon} \int_{\Sigma_\varepsilon} \frac{1}{(1+\beta h)^2}\, dx + \varepsilon C|\Sigma_\varepsilon| \\ &\leq \int_{\partial\Omega} \int_0^1 \frac{\beta^2 h}{(1+\beta h)^2} (1 + \varepsilon\tau hH + \varepsilon^2 Q)\, d\tau\, d\mathcal{H}^{n-1} + \varepsilon C|\Sigma_\varepsilon| \qquad (3.73) \\ &= \int_{\partial\Omega} \frac{\beta^2 h(2 + \varepsilon hH)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1} + O(\varepsilon^2). \end{aligned}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$\varphi_\varepsilon(\sigma + \varepsilon h(\sigma)\nu_0(\sigma)) = \frac{1}{1+\beta h(\sigma)},$$

from which we get

$$\beta \int_{\partial\Omega_\varepsilon} \varphi_\varepsilon^2 \, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} \frac{\beta}{(1+\beta h)^2}(1+\varepsilon h H + \varepsilon^2 Q)\, d\mathcal{H}^{n-1}. \tag{3.74}$$

Finally, joining, (3.73), and (3.74) we have

$$\begin{aligned}\delta\mathcal{E}_{c,\varepsilon}(\varphi_\varepsilon, h) = \frac{\mathcal{E}_{c,\varepsilon}(u_\varepsilon, h) - \mathcal{E}_{c,0}(h)}{\varepsilon} &\leq \int_{\partial\Omega}\left(\frac{\beta^2 h^2 H}{2(1+\beta h)^2} + \frac{\beta h H}{(1+\beta h)^2}\right) d\mathcal{H}^{n-1} + O(\varepsilon)\\ &= \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1} + O(\varepsilon)\end{aligned}$$

so that

$$\limsup_{\varepsilon\to 0^+} \delta\mathcal{E}_{c,\varepsilon}(\varphi_\varepsilon) \leq \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1}$$

and the Γ-limsup inequality is proved. □

3.2.3 Study of the first-order development

Let Ω be a bounded, open set with $C^{1,1}$ boundary. Consider the first-order approximated energy

$$\mathcal{G}_{c,\varepsilon}(\Omega, h) = \beta \int_{\partial\Omega}\left(\frac{1}{1+\beta h} + \varepsilon H \frac{h(2+\beta h)}{2(1+\beta h)^2}\right) d\mathcal{H}^{n-1}.$$

In the following, we drop the dependence on the set Ω and we write $\mathcal{G}_{c,\varepsilon}(h)$ in place of $\mathcal{G}_{c,\varepsilon}(\Omega, h)$. For every $m > 0$, we will consider the problem

$$\inf\left\{ \mathcal{G}_{c,\varepsilon}(h) \mid h \in \mathcal{H}_m \right\}, \tag{3.75}$$

where the set $\mathcal{H}_m$ is defined in (3.41).

Remark 3.21. Despite in Theorem 3.15 we require $h > 0$, it is still meaningful to study the approximating functional $\mathcal{G}_{c,\varepsilon}(h)$, relaxing the constraint to $h \geq 0$. Indeed, even though the positivity of the function h is crucial in the proof of the Theorem 3.15, it is still possible to prove the following: let Ω be a bounded, open set with C^3 boundary, fix a non-negative C^2 function $h\colon \partial\Omega \to [0, +\infty)$, and fix an exponent $\theta \in (1, 2)$; then, carefully retracing all the steps of the proof of Theorem 3.15, we can still prove that the functional $\delta\mathcal{E}_{c,\varepsilon}(\cdot, h + \varepsilon^\theta)$ Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, as $\varepsilon \to 0^+$, to $\mathcal{E}_c^{(1)}(\cdot, h)$, the same functional defined in (3.42).

Assume that a non-zero continuous solution $\mu \in L^1(\partial\Omega)$ to problem (3.75) exists, so that the set $U = \{ \mu > 0 \}$ is open. Then for every $\psi \in C_c^\infty(U)$ with zero mean, and for every $\eta \in \mathbb{R}$ sufficiently small, we can consider the variation $\mu + \eta\psi$ which leads to the Euler-Lagrange equation

$$\int_{\partial\Omega}\left(-\frac{\beta}{(1+\beta\mu)^2} + \frac{\varepsilon H}{1+\beta\mu} - \varepsilon H \frac{\beta\mu(2+\beta\mu)}{(1+\beta\mu)^3}\right)\psi\, d\mathcal{H}^{n-1} = 0.$$

The previous equation yields

$$\frac{c}{\beta}(1+\beta\mu)^3 - (1+\beta\mu) + \frac{\varepsilon H}{\beta} = 0,$$

for some constant $c \in \mathbb{R}$.

Let $\varepsilon > 0$, in the following we will assume that

$$\sup_{\partial\Omega} \frac{\varepsilon H}{\beta} \leq \frac{2}{3}. \tag{3.76}$$

Let

$$H_0 = \inf_{\partial\Omega} H$$

and

$$k_0 = 1 - \frac{\varepsilon H_0}{\beta}.$$

For every $k \in (0, k_0)$, let

$$\Gamma_k = \left\{ \sigma \in \partial\Omega \;\middle|\; \frac{\varepsilon H(\sigma)}{\beta} < 1 - k \right\}$$

and consider

$$P_k(y, \sigma) = ky^3 - y + \frac{\varepsilon H(\sigma)}{\beta}.$$

Notice that, by the choice of k_0, the set Γ_k is always non-empty.

Proposition 3.22. *Let* (3.76) *hold true. Then, for every $k \in (0, k_0)$, and $\sigma \in \Gamma_k$, in the interval $(1, +\infty)$ there exists a unique $y_k(\sigma)$ such that*

$$P_k(y_k(\sigma), \sigma) = 0, \tag{3.77}$$

and there exists $z_k > 1$ such that

$$\max\left\{ \frac{1}{\sqrt{3k}}, 1 \right\} \leq y_k(\sigma) \leq z_k, \tag{3.78}$$

and

$$\lim_{k \to k_0^-} z_k = 1. \tag{3.79}$$

Moreover, for every $k_1 < k_2$ and $\sigma \in \Gamma_{k_2}$ we have that

$$y_{k_2}(\sigma) < y_{k_1}(\sigma). \tag{3.80}$$

Proof. For any fixed $\sigma \in \Gamma_k$ we have that $P_k(1, \sigma) < 0$, and in addition, for $k \geq 1/3$, the polynomial $P_k(y, \sigma)$ is strictly increasing in $y \geq 1$, while for $k < 1/3$ we have that

$$\frac{\partial}{\partial y} P_k(y, \sigma) < 0 \qquad \text{if } y \in \left[1, \frac{1}{\sqrt{3k}}\right),$$
$$\frac{\partial}{\partial y} P_k(y, \sigma) > 0 \qquad \text{if } y \in \left(\frac{1}{\sqrt{3k}}, +\infty\right).$$

Therefore, in the interval $(1, +\infty)$ there exists a unique zero $y_k(\sigma)$ of the polynomial $P_k(\cdot, \sigma)$, and

$$y_k(\sigma) \geq \frac{1}{\sqrt{3k}}.$$

Notice in addition that for every $y > 1$, we have that $y < y_k(\sigma)$ if and only if $P_k(y, \sigma) < 0$. Hence, if we choose z_k to be the unique real number in $(1, +\infty)$ such that

$$kz_k^3 - z_k + \frac{\varepsilon H_0}{\beta} = 0, \tag{3.81}$$

then

$$P_k(z_k,\sigma) = \frac{\varepsilon H(\sigma)}{\beta} - \frac{\varepsilon H_0}{\beta} \geq 0,$$

and we have that (3.78) holds.

We now prove (3.79). We first observe that z_k is decreasing in k: let $k_1 < k_2$, so that

$$k_1 z_{k_2}^3 - z_{k_2} + \frac{\varepsilon H_0}{\beta} < k_2 z_{k_2}^3 - z_{k_2} + \frac{\varepsilon H_0}{\beta} = 0 = k_1 z_{k_1}^3 - z_{k_1} + \frac{\varepsilon H_0}{\beta},$$

which ensures

$$z_{k_2} < z_{k_1},$$

indeed the polynomial $k_1 y^3 - y + \varepsilon H_0/\beta$ is negative on $(1, z_{k_1})$ and it is non-negative on $[z_{k_1}, +\infty)$. We now have that there exists

$$z = \lim_{k \to k_0^-} z_k,$$

and, passing to the limit in (3.81) and recalling that by definition $\beta k_0 = \varepsilon H_0$, we get that z solves the equation

$$k_0(z^3 - 1) - z + 1 = 0.$$

From (3.76), we have that $k_0 > 1/3$, so that $z = 1$ is the unique solution in $[1, +\infty)$ to the previous equation, proving (3.79).

Finally, in order to prove (3.80), let $k_1 < k_2$ and $\sigma \in \Gamma_{k_1} \cap \Gamma_{k_2} = \Gamma_{k_2}$, then (3.77) ensures that

$$P_{k_1}(y_{k_2},\sigma) < P_{k_2}(y_{k_2},\sigma) = 0,$$

from which

$$y_{k_2}(\sigma) < y_{k_1}(\sigma).$$

□

Let $k \in (0, k_0)$, and let y_k be as in Proposition 3.22, we define

$$\mu_k(\sigma) = \begin{cases} \dfrac{1}{\beta}(y_k(\sigma) - 1) & \text{if } \sigma \in \Gamma_k. \\[2ex] 0 & \text{if } \sigma \in \partial\Omega \setminus \Gamma_k, \end{cases}$$

Notice that, by (3.80), μ_k is decreasing in k.

We have the following

Proposition 3.23. *Let* (3.76) *hold true. Then, for every* $m > 0$ *,there exists a unique* $k = k_m \in (0, k_0)$ *such that*

$$\int_{\partial\Omega} \mu_k \, d\mathcal{H}^{n-1} = m.$$

Proof. We first prove that the function

$$M(k) = \int_{\partial\Omega} \mu_k \, d\mathcal{H}^{n-1}$$

is continuous. Fix $k \in (0, k_0)$ and let $\delta > 0$, then

$$\beta(M(k) - M(k+\delta)) = \int_{\Gamma_{k+\delta}} (y_k - y_{k+\delta}) \, d\mathcal{H}^{n-1} + \int_{\left\{1-k-\delta \leq \frac{\varepsilon H}{\beta} < 1-k\right\}} (y_k - 1) \, d\mathcal{H}^{n-1}. \quad (3.82)$$

By definition, we have that for every $\sigma \in \partial\Omega$

$$\lim_{\delta\to 0^+} \chi_{\Gamma_{k+\delta}}(\sigma) = \chi_{\Gamma_k}(\sigma).$$

Let $\sigma \in \Gamma_k$, then the function $y_{k+\delta}(\sigma)$ is defined for small enough $\delta < \delta_\sigma$, and by the implicit function theorem and the regularity of $P_k(y, \sigma)$, we get

$$\lim_{\delta\to 0^+} y_{k+\delta}(\sigma) = y_k(\sigma).$$

Therefore, by (3.78), the monotonicity of z_k, and the dominated convergence theorem we have

$$\lim_{\delta\to 0^+} \int_{\Gamma_{k+\delta}} (y_k - y_{k+\delta})\, d\mathcal{H}^{n-1} = 0. \tag{3.83}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$\lim_{\delta\to 0^+} \chi_{\left\{\, 1-k-\delta \le \frac{\varepsilon H}{\beta} < 1-k \,\right\}}(\sigma) = 0,$$

which entails

$$\lim_{\delta\to 0^+} \int_{\left\{\, 1-k-\delta \le \frac{\varepsilon H}{\beta} < 1-k \,\right\}} (y_k - 1)\, d\mathcal{H}^{n-1} = 0. \tag{3.84}$$

Joining (3.82), (3.83), and (3.84), we get

$$\lim_{\delta\to 0^+} M(k) - M(k+\delta) = 0.$$

We now fix $k \in (0, k_0)$, $\delta > 0$, and we compute

$$\beta(M(k-\delta) - M(k)) = \int_{\Gamma_k} (y_{k-\delta} - y_k)\, d\mathcal{H}^{n-1} + \int_{\left\{\, 1-k \le \frac{\varepsilon H}{\beta} < 1-k+\delta \,\right\}} (y_{k-\delta} - 1)\, d\mathcal{H}^{n-1}. \tag{3.85}$$

As in the previous case, by the implicit function theorem, for every $\sigma \in \Gamma_k$,

$$\lim_{\delta\to 0^+} y_{k-\delta}(\sigma) = y_k(\sigma).$$

By the dominated convergence theorem,

$$\lim_{\delta\to 0^+} \int_{\Gamma_k} (y_{k-\delta} - y_k)\, d\mathcal{H}^{n-1} = 0. \tag{3.86}$$

On the other hand, we have that for every $\sigma \in \partial\Omega$

$$\lim_{\delta\to 0^+} \chi_{\left\{\, 1-k \le \frac{\varepsilon H}{\beta} < 1-k+\delta \,\right\}}(\sigma) = \chi_{\left\{\, \frac{\varepsilon H}{\beta} = 1-k \,\right\}}(\sigma),$$

and, for every σ such that $\varepsilon H(\sigma) = \beta(1-k)$, we may use the monotonicity of $y_{k-\delta}$ and then we pass to the limit in (3.77), having that

$$\lim_{\delta\to 0^+} y_{k-\delta}(\sigma) = 1,$$

which entails

$$\lim_{\delta\to 0^+} \int_{\left\{\, 1-k \le \frac{\varepsilon H}{\beta} < 1-k+\delta \,\right\}} (y_{k-\delta} - 1)\, d\mathcal{H}^{n-1} = 0. \tag{3.87}$$

Joining (3.85), (3.86), and (3.87), we get

$$\lim_{\delta\to 0^+} M(k-\delta) - M(k) = 0,$$

thus concluding the proof of the continuity of M.

Finally, by monotonicity, and by (3.78), and (3.76), we have that

$$\lim_{k\to 0^+} M(k) = +\infty, \qquad \lim_{k\to k_0^-} M(k) = 0.$$

and the proposition is proved. $\square$

Theorem 3.24. *Let* (3.76) *hold true. Then, for every* $m > 0$*, the function* μ_{k_m} *is the unique minimiser to problem* (3.75)*.*

Proof. Let $h\colon \partial\Omega \to R$ with $h \geq 0$ and

$$\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} \leq m.$$

For every $t \in [0,1]$ consider

$$h_t = \mu_k + t(h - \mu_k)$$

and

$$g(t) = \mathcal{G}_{c,\varepsilon}(h_t).$$

We claim that $g(t)$ is increasing in t. By explicit computation, we have that

$$g'(t) = \beta \int_{\partial\Omega} \frac{\varepsilon H - \beta(1+\beta h_t)}{(1+\beta h_t)^3}(h - \mu_k) \, d\mathcal{H}^{n-1}.$$

From (3.76) we have that, for every $\sigma \in \partial\Omega$ the function

$$x \mapsto \frac{\varepsilon H - \beta(1+\beta x)}{(1+\beta x)^3}$$

is increasing on $[0, +\infty)$, so that

$$g'(t) \geq \beta \int_{\partial\Omega} \frac{\varepsilon H - \beta(1+\beta\mu_k)}{(1+\beta\mu_k)^3}(h - \mu_k) \, d\mathcal{H}^{n-1}.$$

Notice that for every $\sigma \in \partial\Omega$

$$\frac{\varepsilon H - \beta(1+\beta\mu_k)}{(1+\beta\mu_k)^3}(h - \mu_k) \geq -\beta k(h - \mu_k).$$

Indeed, if $\sigma \in \Gamma_k$, then by (3.77),

$$\varepsilon H(\sigma) - \beta(1 + \beta\mu_k(\sigma)) = -\beta k(1 + \beta\mu_k(\sigma))^3$$

and equality holds. On the other hand, if $\sigma \in \partial\Omega \setminus \Gamma_k$, then $\mu_k(\sigma) = 0$ and $\varepsilon H(\sigma) \geq \beta(1-k)$, so that

$$\begin{aligned}
\frac{\varepsilon H(\sigma) - \beta(1+\beta\mu_k(\sigma))}{(1+\beta\mu_k(\sigma))^3}(h(\sigma) - \mu_k(\sigma)) &= (\varepsilon H(\sigma) - \beta)h(\sigma) \\
&\geq -\beta k h(\sigma) \\
&= -\beta k(h(\sigma) - \mu_k(\sigma)).
\end{aligned}$$

Then we have

$$g'(t) \geq \beta^2 k \int_{\partial\Omega} (\mu_k - h) \, d\mathcal{H}^{n-1} \geq 0,$$

and the claim is proven. In particular, we have that

$$\mathcal{G}_{c,\varepsilon}(\mu_k) = g(0) \leq g(1) = \mathcal{G}_{c,\varepsilon}(h)$$

that is, μ_k is a minimiser for problem (3.75). Finally, by (3.76), we have that, for every $\sigma \in \partial\Omega$, the function

$$x \in [0, +\infty) \mapsto \frac{1}{1+\beta x} + \varepsilon H(\sigma)\frac{x(2+\beta x)}{2(1+\beta x)^2}$$

is strictly convex, thus problem (3.75) admits a unique minimiser. □

Remark 3.25. Notice that the optimal configuration μ_{k_m} concentrates where the mean curvature is smaller: for simplicity, let us write $k = k_m$, and let us take $\sigma_1, \sigma_2 \in \Gamma_k$ such that

$$H(\sigma_1) < H(\sigma_2).$$

Noticing that

$$P_k(y_k(\sigma_1), \sigma_1) = 0 = P_k(y_k(\sigma_2), \sigma_2),$$

we get

$$ky_k(\sigma_1)^3 - y_k(\sigma_1) > ky_k(\sigma_2)^3 - y_k(\sigma_2).$$

By (3.78), we can use the monotonicity of the function $y \mapsto ky^3 - y$ on $[1/\sqrt{3k}, +\infty)$, getting

$$y_k(\sigma_1) > y_k(\sigma_2),$$

so that $\mu_k(\sigma_1) > \mu_k(\sigma_2)$.

Remark 3.26. Notice that, if Ω is a ball, we have that the mean curvature is constant, so that $\Gamma_k = \partial\Omega$, and μ_k is a constant function. Hence, if (3.76) holds, problem (3.75) admits as unique solution the constant function

$$\mu = \frac{m}{P(\Omega)}.$$

Remark 3.27. Notice that, if for every $\sigma \in \partial\Omega$

$$\frac{\varepsilon H(\sigma)}{\beta} \geq 2, \tag{3.88}$$

then, the optimal configuration is given by $\mu \equiv 0$. Indeed if (3.88) holds, for every $\sigma \in \partial\Omega$ the function

$$x \in [0, +\infty) \mapsto \frac{1}{1 + \beta x} + \frac{\varepsilon H(\sigma)}{\beta} \frac{\beta x(2 + \beta x)}{2(1 + \beta x)^2}$$

reaches its minimum for $x = 0$.

3.3 Related problems in shape optimization

In the previous sections, we studied the energy $\mathcal{E}_c$ when the set A varies and the set Ω is fixed. However, it is natural to ask what happens when the set Ω is allowed to vary. Namely, we pose the question of which is the best (or worst) shape for the conductor Ω. To investigate such a question, we will explicitly write the dependence on the set Ω as $\mathcal{E}_c(\Omega; A)$. For every compact set $\Omega \subset \mathbb{R}^n$ and open set A with Lipschitz boundary such that $\Omega \subset A$ denote by

$$\operatorname{Cap}_\beta(\Omega; A) = \inf \left\{ \int_A |\nabla v|^2 \, dx + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} \;\middle|\; \begin{array}{c} v \in H^1(A), \\ v = 1 \text{ in } \Omega \end{array} \right\}.$$

the relative Robin capacity of Ω with respect to A, so that

$$\mathcal{E}_c(\Omega; A) = \operatorname{Cap}_\beta(\Omega; A) + C_0 \mathcal{L}^n(A \setminus \Omega).$$

We will be interested in minimisation (or maximisation) of Cap_β when Ω and A vary in appropriate classes of sets.

Minimisation under volume constraints

The most intuitive problem to study is probably the minimisation problem under volume constraint. That is, let ω_n be the volume of the unit ball in $\mathbb{R}^n$ and let $R > 1$, consider the problem

$$\inf \left\{ \operatorname{Cap}_\beta(\Omega; A) \;\middle|\; \mathcal{L}^n(\Omega) = \omega_n,\, \Omega \subset A,\, \mathcal{L}^n(A) \le \omega_n R^n \right\}. \tag{3.89}$$

If we consider the analogous problem in the case of the relative Dirichlet capacity

$$\operatorname{Cap}(\Omega; A) = \inf \left\{ \int_A |\nabla v|^2 \, dx \;\middle|\; v \in H^1_0(A)\, v \ge 1 \text{ in } \Omega \right\},$$

by the Pólya-Szegő inequality for the Schwarz rearrangement (see, for instance, [58, 49]), we immediately have that

$$\inf \{ \operatorname{Cap}(\Omega; A) \mid \mathcal{L}^n(\Omega) = \omega_n,\, \Omega \subset A,\, \mathcal{L}^n(A) \le \omega_n R^n \} = \operatorname{Cap}(B_1, B_R),$$

where B_1, B_R are two concentric balls of radii 1 and R. Then, it is natural to expect a similar result also for problem (3.89). In [25] (see also [2]), the authors prove the following theorem.

Theorem 3.28. *The solution to problem* (3.89) *is given by two concentric balls* (B_1, B_r). *Moreover we have that*

- *if* $\beta \ge n-1$, *then* $r = R$;
- *if* $n-2 < \beta < n-1$, *then there exists a unique* $R_\beta \ge \frac{n-1}{\beta}$, *such that*
  - *if* $R_\beta < R$, *then* $r = R$,
  - *if* $R_\beta = R$, *then* $r = 1$ *and* $r = R$ *are both solutions,*
  - *if* $R_\beta > R$, *then* $r = 1$;
- *if* $0 < \beta \le n-2$, *then* $r = 1$.

It is worth noticing that, at least when $\beta > n-2$, the proof of Theorem 3.28 follows the strategy of the original proof of the Bossel-Daners inequality.

Let $\Theta \colon [0,1] \to \mathbb{R}^+$ be a lower semicontinous, non decreasing function such that $\Theta(0) = 0$, and let

$$\operatorname{Cap}_\Theta(\Omega; A) = \inf \left\{ \int_A |\nabla v|^2 \, dx + \int_{\partial A} \Theta(v) \, d\mathcal{H}^{n-1} \;\middle|\; \begin{array}{c} v \in H^1(A), \\ v \in [0,1] \\ v = 1 \text{ in } \Omega \end{array} \right\}.$$

In [25], the authors also prove the following theorem.

Theorem 3.29. *The problem*

$$\min_{\Omega \subset A, \mathcal{L}^n(\Omega) = \omega_n} \operatorname{Cap}_\Theta(\Omega; A) + C_0 \mathcal{L}^n(A \setminus \Omega),$$

admits two concentric balls as a solution.

Recall the definition of the first-order approximated energy

$$\mathcal{G}_{c,\varepsilon}(\Omega, h) = \beta \int_{\partial\Omega} \left(\frac{1}{1+\beta h} + \varepsilon H_\Omega \frac{h(2+\beta h)}{2(1+\beta h)^2} \right) d\mathcal{H}^{n-1},$$

where H_Ω is the mean curvature of the boundary of Ω, and let

$$\mathcal{G}_{c,\varepsilon}(\Omega) = \inf_{h \in \mathcal{H}_m} \mathcal{G}_{c,\varepsilon}(\Omega, h).$$

Theorem 3.28 naturally leads to the following question

Open problem 1. *Prove or disprove that the problem*

$$\inf \left\{ \mathcal{G}_\varepsilon(\Omega, h) \,\middle|\, \begin{array}{l} |\Omega| = \omega_n, \\ h \in \mathcal{H}_m \end{array} \right\}$$

admits the couple (B_1, h^*) *as a solution, where* h^* *is a constant function.*

Maximisation under various geometric constraints

Let $\delta > 0$ and let

$$\mathcal{I}_\delta(\Omega) = \operatorname{Cap}_\beta(\Omega; \Omega + \delta B_1),$$

where $B_1 \subset \mathbb{R}^n$ is the unit ball, and

$$\Omega + \delta B_1 = \{\, x + \delta y \colon x \in \Omega,\, y \in B_1 \,\},$$

denotes the Minkowski sum of Ω and δB_1. In [37] (see also [12]), the authors study the maximisation of $\mathcal{I}_\delta$ under different geometric constraints depending on the dimension n. In particular, when $n = 2$, they prove the following

Theorem 3.30. *Let* $\Omega \subset \mathbb{R}^2$ *be an open, bounded, connected set with piecewise* C^1 *boundary. Then*

$$\mathcal{I}_\delta(\Omega) \le \mathcal{I}_\delta(\Omega^*)$$

where Ω^* *is the disk having the same perimeter as* Ω.

To generalise the previous theorem to higher dimensions, we define the *quermassintegral*. We refer to [60, 26] for the following

Definition 3.31 (Quermassintegrals)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty, bounded, convex set. We define the *quermassintegrals* as the unique coefficients $W_j(\Omega)$ such that

$$\mathcal{L}^n(\Omega + tB_1) = \sum_{j=0}^{n} \binom{n}{j} W_j(\Omega) t^j.$$

In particular $W_0(\Omega)$ is the measure of Ω and $W_n(\Omega) = \omega_n$, the measure of the unit ball.

The following theorem can be seen as a generalisation of the isoperimetric inequality for convex sets.

Theorem 3.32 (Alexandrov-Fenchel Inequality)**.** *Let* $0 \le i < j \le n - 1$*, and let* $\Omega \subseteq \mathbb{R}^n$ *be a non-empty, bounded, convex set, then*

$$\left(\frac{W_i(\Omega)}{\omega_n} \right)^{\frac{1}{n-i}} \le \left(\frac{W_j(\Omega)}{\omega_n} \right)^{\frac{1}{n-j}}.$$

Moreover, the inequality holds as equality if and only if Ω *is a ball.*

Remark 3.33. Let Ω be a bounded, open, convex set with C^2 boundary and non-zero Gaussian curvature, then the quermassintegrals are related to the principal curvatures of the boundary of Ω. Indeed we have that for every $j = 1, \dots, n$

$$W_j = \frac{1}{n} \int_{\partial\Omega} H_{j-1}(\sigma) \, d\mathcal{H}^{n-1}$$

Here H_j denotes the j-th normalised elementary symmetric function of the principal curvatures of $\partial\Omega$, that is $H_0 = 1$ and, for every $j = 1, \dots n-1$,

$$H_j(\sigma) = \binom{n-1}{j}^{-1} \sum_{1 \le i_1 \le \dots \le i_j \le n-1} k_{i_1}(\sigma) \cdots k_{i_j}(\sigma),$$

where $k_1(\sigma), \dots, k_{n-1}(\sigma)$ are the principal curvatures at a point $\sigma \in \partial\Omega$. In particular, we have that

$$W_1(\Omega) = \frac{1}{n} P(\Omega)$$

and

$$W_2(\Omega) = \frac{1}{n(n-1)} \int_{\partial\Omega} H_\Omega \, d\mathcal{H}^{n-1}.$$

In [37] (see also [12]), the authors prove the following theorem.

Theorem 3.34. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, convex set. Then*

$$\mathcal{I}_\delta(\Omega) \le \mathcal{I}_\delta(\Omega^*)$$

where Ω^* is the ball having the same W_{n-1} quermassintegral as Ω.

Similar results also apply to the approximated energy $\mathcal{G}_{c,\varepsilon}$. Indeed, in [5], we prove the following propositions.

Proposition 3.35. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, simply connected set with C^2 boundary such that*

$$P(\Omega) \ge 3\pi \frac{\varepsilon}{\beta}.$$

Then

$$\mathcal{G}_{c,\varepsilon}(\Omega) \le \mathcal{G}_{c,\varepsilon}(\Omega^*), \tag{3.90}$$

where Ω^* is the disk having the same perimeter as Ω.

Proposition 3.36. *Let $n \ge 3$, $2 \le k \le n-1$, and let $\Omega \subset \mathbb{R}^n$ be a bounded, open, convex set with C^2 boundary and non zero Gaussian curvature such that*

$$W_k(\Omega) \ge \omega_n \left(\frac{3(n-1)\varepsilon}{2\beta} \right)^{n-k} \tag{3.91}$$

Then

$$\mathcal{G}_{c,\varepsilon}(\Omega) \le \mathcal{G}_{c,\varepsilon}(\Omega^*)$$

where Ω^* is the ball having the same W_k quermassintegral as Ω.

We only prove Proposition 3.36 as the proof of Proposition 3.35 is analogous.

Proof. From the Alexandrov-Fenchel inequalities, as $W_k(\Omega) = W_k(\Omega^*)$, we have that for every $0 \le i \le k$

$$\left(\frac{W_i(\Omega^*)}{\omega_n} \right)^{\frac{1}{n-i}} = \left(\frac{W_k(\Omega^*)}{\omega_n} \right)^{\frac{1}{n-k}} = \left(\frac{W_k(\Omega)}{\omega_n} \right)^{\frac{1}{n-k}} \ge \left(\frac{W_i(\Omega)}{\omega_n} \right)^{\frac{1}{n-i}},$$

that is

$$W_i(\Omega) \le W_i(\Omega^*).$$

In particular, we have that

$$P(\Omega) \le P(\Omega^*)$$

and

$$\int_{\partial\Omega} H_\Omega \, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega^*} H_{\Omega^*} \, d\mathcal{H}^{n-1}.$$

Moreover, since

$$\left(\frac{W_k(\Omega^*)}{\omega_n}\right)^{\frac{1}{n-k}} = \left(\frac{W_1(\Omega^*)}{\omega_n}\right)^{\frac{1}{n-1}} = \left(\frac{P(\Omega^*)}{n\omega_n}\right)^{\frac{1}{n-1}} = \frac{n-1}{H_{\Omega^*}},$$

condition 3.91 read as

$$\frac{\varepsilon H_{\Omega^*}}{\beta} \leq \frac{2}{3}.$$

by Remark 3.26, we get

$$\mathcal{G}_{c,\varepsilon}(\Omega^*) = \mathcal{G}_{c,\varepsilon}(\Omega^*, m/P(\Omega^*)).$$

On the other hand,

$$\begin{aligned}
\mathcal{G}_{c,\varepsilon}(\Omega) &\leq \mathcal{G}_{c,\varepsilon}(\Omega, m/P(\Omega)) \\
&= \beta\left(\frac{P(\Omega)^2}{P(\Omega)+\beta m} + \frac{\varepsilon m(2P(\Omega)+\beta m)}{2(P(\Omega)+\beta m)^2}\int_{\partial\Omega} H_\Omega \, d\mathcal{H}^{n-1}\right) \\
&\leq \beta\left(\frac{P(\Omega^*)^2}{P(\Omega^*)+\beta m} + \frac{\varepsilon m(2P(\Omega^*)+\beta m)}{2(P(\Omega^*)+\beta m)^2}\int_{\partial\Omega^*} H_{\Omega^*} \, d\mathcal{H}^{n-1}\right) = \mathcal{G}_{c,\varepsilon}(\Omega^*).
\end{aligned}$$

□

Minimisation under perimeter constraint.

In [37], the authors pose the question of finding the infimum of $\mathcal{I}_\delta$ among convex sets of given perimeter. In [5], we prove the following theorem, which may suggest that the infimum of $\mathcal{I}_\delta$ is asymptotically achieved by a sequence of thinning sets collapsing to a flat $(n-1)$-dimensional disk.

For every $P > 0$ let

$$\mathcal{K}_P = \left\{ \Omega \subset \mathbb{R}^n \;\middle|\; \begin{array}{l} \Omega \text{ open and bounded with } C^{1,1} \text{ boundary} \\ H_\Omega \geq 0 \\ P(\Omega) = P \end{array} \right\},$$

we consider the problem

$$\inf_{\Omega \in \mathcal{K}_P} \mathcal{G}_{c,\varepsilon}(\Omega).$$

Definition 3.37 (Cookie Shape)**.** For any $r, R > 0$ we define the *cookie shape*

$$C_{r,R} = \left\{ (x', x_n) \;\middle|\; -f_{r,R}(x') \leq x_n \leq f_{r,R}(x') \right\},$$

where

$$f_{r,R}(x') = \begin{cases} r & |x'| \leq R, \\ \sqrt{r^2 - (|x'|-R)^2} & R < |x'| \leq R + r. \end{cases}$$

We have the following theorem.

Theorem 3.38. *For every $P, m > 0$ we have*

$$\inf_{\Omega \in \mathcal{K}_P} \mathcal{G}_{c,\varepsilon}(\Omega) = \frac{\beta P^2}{P + \beta m}$$

and the infimum is asymptotically achieved by a sequence of thin cookie shapes.

Proof. The inequality

$$\inf_{\Omega \in \mathcal{K}_P} \mathcal{G}_{c,\varepsilon}(\Omega) \geq \frac{\beta P^2}{P + \beta m}$$

is a straightforward consequence of Proposition 3.18 and the fact that $H_\Omega \geq 0$. To prove the reverse inequality, let $r_k > 0$ be a decreasing sequence such that

$$\lim_k r_k = 0.$$

Then we can find an increasing sequence R_k such that the thinning sequence of cookie shapes $C_{r_k,R_k} \in \mathcal{K}_p$ and

$$\lim_k R_k = \left(\frac{P}{2\omega_{n-1}}\right)^{\frac{1}{n-1}}.$$

For every k let

$$h_k(\sigma) = \begin{cases} \dfrac{m}{2\omega_{n-1}R_k^{n-1}} & \text{if } H_{C_{r_k,R_k}}(\sigma) = 0, \\ 0 & \text{if } H_{C_{r_k,R_k}}(\sigma) > 0. \end{cases}$$

Then

$$\mathcal{G}_{c,\varepsilon}(C_{r_k,R_k}) \leq \mathcal{G}_{c,\varepsilon}(C_{r_k,R_k}, h_k),$$

and by direct computation, one can prove that

$$\lim_k \mathcal{G}_{c,\varepsilon}(C_{r_k,R_k}, h_k) = \frac{\beta P^2}{P + \beta m},$$

which concludes the proof. □

Chapter 4

The Poisson Problem

In this chapter, we will study the optimisation problem of finding the best configuration of insulating material surrounding a heated conductor. In particular, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded open set, representing the conductor, let $f \in L^2(\Omega)$ be a positive function representing the heat source, and let A be a set containing Ω, such that the set $\Sigma = A \setminus \overline{\Omega}$ represents the configuration of insulating material. The steady-state temperature of the configuration is given by

$$\begin{cases} -\Delta u_A = f & \text{in } \Omega, \\ \Delta u_A = 0 & \text{in } A \setminus \overline{\Omega}, \\ \partial_{\nu_0} u_A^- = k \partial_{\nu_0} u_A^+ & \text{on } \partial\Omega \\ k\partial_\nu u_A + \beta u_A = 0 & \text{on } \partial A, \end{cases} \tag{4.1}$$

where ν_0 and ν are the outer unit normal to the boundary of Ω and A respectively, $\partial_{\nu_0} u^-$ and $\partial_{\nu_0} u^+$ are the normal derivative on $\partial\Omega$ from inside Ω and from inside $A \setminus \overline{\Omega}$ respectively, and β is a positive parameter. Then the best configuration is the one that minimises the energy

$$\mathcal{E}_p(A) = \int_\Omega |\nabla u_A|^2 \, dx + k \int_{A\setminus\overline{\Omega}} |\nabla u_A|^2 \, dx + \beta \int_{\partial A} u_A^2 \, d\mathcal{H}^{n-1} - 2 \int_\Omega f u_A \, dx + C_0 \mathcal{L}^n\big(A \setminus \overline{\Omega}\big). \tag{4.2}$$

This chapter is structured as follows. In Section 4.1 we prove existence and regularity results for the minimiser of the problem. In Section 4.2 we study the limit problem in the thin layer setting, and characterise its solution. In Section 4.3 we prove a first-order development of the energy in the thin layer setting.

4.1 Bulk insulating layer: existence and regularity

The content of this section is based on the results of [4]. For simplicity's sake, we will assume $k = 1$.

Our aim is to minimise the energy

$$\mathcal{E}_p(A, v) = \int_A |\nabla v|^2 \, dx + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} - 2 \int_\Omega f v \, dx + C_0 \mathcal{L}^n\big(A \setminus \overline{\Omega}\big),$$

where A is open set with Lipschitz boundary containing Ω and $v \in H^1(A)$. As we did in Section 3.1, for the capacitary problem, we will identify the set A with the set $\{\, v > 0 \,\}$ and extend the function v to be equal to zero outside of A. Such an extension of the function v is in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$ and the energy can be written as

$$\mathcal{E}_p(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, dx + \beta \int_{J_v} \big(\overline{v}^2 + \underline{v}^2\big) \, d\mathcal{H}^{n-1} - 2 \int_\Omega f v \, dx + C_0 \mathcal{L}^n(\{\, v > 0 \,\} \setminus \Omega), \tag{4.3}$$

where ∇v is the absolutely continuous part of the distributional gradient of v, J_v is the jump set of v, and $\overline{v}$ and $\underline{v}$ are the approximate upper and lower limits respectively.

Then the minimisation problem is

$$\inf\left\{\,\mathcal{E}_p(v)\;\middle|\;v\in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)\cap H^1(\Omega)\,\right\}. \tag{4.4}$$

We summarise the main results of this section in the following theorems.

Theorem 4.1. *Let $n\geq 2$, let $\Omega\subset\mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary, let $f\in L^2(\Omega)$, with $f>0$ almost everywhere in Ω. Assume in addition that, if $n=2$,*

$$\|f\|_{2,\Omega}^2 < C_0\lambda_\beta(B)\mathcal{L}^2(\Omega), \tag{4.5}$$

where B is a ball having the same measure as Ω. Then problem (4.4) admits a solution. Moreover, if $p>n$ and $f\in L^p(\Omega)$, then there exists a positive constant $C=C(\Omega,f,p,\beta,C_0)$ such that if u is a minimiser to problem (4.4) then

$$\|u\|_\infty \leq C.$$

Theorem 4.2. *Let $n\geq 2$, let $\Omega\subset\mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary, let $p>n$ and let $f\in L^p(\Omega)$, with $f>0$ almost everywhere in Ω. Assume in addition that, if $n=2$, condition (4.5) holds. Then there exist positive constants $\delta_0=\delta_0(\Omega,f,p,\beta,C_0)$, $c=c(\Omega,f,p,\beta,C_0)$, $C=C(\Omega,f,p,\beta,C_0)$ such that if u is a minimiser to problem (4.4) then*

$$u\geq\delta_0 \qquad \mathcal{L}^n\text{-a.e. in } \{\,u>0\,\},$$

and the jump set J_u satisfies the density estimates

$$cr^{n-1}\leq\mathcal{H}^{n-1}(J_u\cap B_r(x))\leq Cr^{n-1},$$

with $x\in\overline{J_u}$, and $0<r<d(x,\partial\Omega)$. In particular, we have

$$\mathcal{H}^{n-1}(\overline{J_u}\setminus J_u)=0.$$

subsection 4.1.1 is devoted to the proof of Theorem 4.1, while subsection 4.1.2 is devoted to the proof of Theorem 4.2.

Remark 4.3. We remark that, as is well known (see for instance [43, Theorem 8.15]), in the regular case, to ensure the boundedness of solutions to equation (4.1) is enough to assume $f\in L^p(\Omega)$ with $p>n/2$. However, our assumption $p>n$ is coherent with the results about the generalised solutions for elliptic Robin boundary value problems on arbitrary domains by [35] (see also [32]).

4.1.1 Existence of minimisers

In this subsection, we prove Theorem 4.1: in Proposition 4.7 we prove the existence of a minimiser to problem (4.4); in Proposition 4.11 we prove the L^∞ estimate for a minimiser.

We will assume that $\Omega\subseteq\mathbb{R}^n$ is an open bounded set with $C^{1,1}$ boundary, that $f\in L^2(\Omega)$ is a positive function and that β, C_0 are positive constants. We consider the energy functional $\mathcal{E}_p$ defined in (4.3).

We will need the following simple lemma, whose proof we omit.

Lemma 4.4. *For every $0<r<R$, the following inequality holds*

$$\lambda_\beta(B_r)\leq\left(\frac{\mathcal{L}^n(B_R)}{\mathcal{L}^n(B_r)}\right)^{\frac{2}{n}}\lambda_\beta(B_R),$$

where B_R and B_r are balls with radii R and r respectively.

We can now prove the following lemma, which will be crucial in proving the existence.

Lemma 4.5. *Let $n \geq 2$ and assume that, if $n = 2$, condition (4.5) holds true. Then there exist two positive constants $c = c(\Omega, f, \beta, C_0)$ and $C = C(\Omega, f, \beta, C_0)$ such that if $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$, with $\mathcal{E}_p(v) \leq 0$ and $\Omega \subseteq \{ v > 0 \}$, then*

$$\mathcal{L}^n(\{ v > 0 \}) \leq c, \tag{4.6}$$

$$\|v\|_2 \leq C. \tag{4.7}$$

Proof. Let B' be a ball with the same measure as $\{ v > 0 \}$. By the Poincaré-type inequality (1.11)

$$\begin{aligned} 0 \geq \mathcal{E}_p(v) \geq \ \lambda_\beta(B') \int_{\mathbb{R}^n} v^2 \, dx - 2 \int_\Omega f v \, dx \\ + C_0 \mathcal{L}^n(\{ v > 0 \} \setminus \Omega). \end{aligned}$$

By Lemma 4.4 and Hölder inequality

$$\begin{aligned} 0 \geq & \lambda_\beta(B) \left(\frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\{ v > 0 \})} \right)^{\frac{2}{n}} \|v\|_2^2 - 2\|f\|_{2,\Omega} \|v\|_2 \\ & + C_0 \mathcal{L}^n(\{ v > 0 \} \setminus \Omega) \end{aligned} \tag{4.8}$$

where B is a ball with the same measure as Ω. Obviously (4.8) implies that

$$\|f\|_{2,\Omega}^2 - \lambda_\beta(B) \left(\frac{\mathcal{L}^n(\Omega)}{\mathcal{L}^n(\{ v > 0 \})} \right)^{\frac{2}{n}} C_0 \mathcal{L}^n(\{ v > 0 \} \setminus \Omega) \geq 0.$$

Let $M = \mathcal{L}^n(\{ v > 0 \})$, and notice that, since $\Omega \subseteq \{ v > 0 \}$,

$$\mathcal{L}^n(\{ v > 0 \} \setminus \Omega) = M - \mathcal{L}^n(\Omega),$$

therefore

$$\|f\|_{2,\Omega}^2 \geq C_0 \lambda_\beta(B) (\mathcal{L}^n(\Omega))^{\frac{2}{n}} \left(M^{1-\frac{2}{n}} - M^{-\frac{2}{n}} \mathcal{L}^n(\Omega) \right).$$

This implies (taking into account (4.5) if $n = 2$) that there exists $c = c(\Omega, f, \beta, C_0) > 0$ such that

$$\mathcal{L}^n(\{ v > 0 \}) < c.$$

Finally observe that by (4.8) it follows

$$\|v\|_2 \leq C(M), \tag{4.9}$$

where

$$\begin{aligned} C(M) = & \frac{M^{\frac{2}{n}} \left(\|f\|_{2,\Omega} + \sqrt{\|f\|_{2,\Omega}^2 - C_0 \lambda_\beta(B) \left(\frac{\mathcal{L}^n(\Omega)}{M} \right)^{\frac{2}{n}} (M - \mathcal{L}^n(\Omega))} \right)}{\lambda_\beta(B) \mathcal{L}^n(\Omega)} \\ \leq & \frac{2c^{\frac{2}{n}} \|f\|_{2,\Omega}}{\lambda_\beta(B) \mathcal{L}^n(\Omega)} \end{aligned}$$

□

Remark 4.6. Let $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$, it is always possible to choose a function v_0 such that $v_0 = v$ in $\mathbb{R}^n \setminus \Omega$, $\mathcal{E}_p(v_0) \leq \mathcal{E}_p(v)$, and $\Omega \subseteq \{ v_0 > 0 \}$. Indeed the function $v_0 \in H^1(\Omega)$, weak solution to the following boundary value problem

$$\begin{cases} -\Delta v_0 = f & \text{in } \Omega, \\ v_0 = \gamma^-_{\partial\Omega}(v) & \text{on } \partial\Omega, \end{cases} \tag{4.10}$$

satisfies

$$\int_\Omega \nabla v_0 \cdot \nabla \varphi \, dx = \int_\Omega f \varphi \, dx$$

for every $\varphi \in H^1_0(\Omega)$ and $v_0 = \gamma^-_{\partial\Omega}(v)$ on $\partial\Omega$ in the sense of the trace. Then, extending v_0 to be equal to v outside of Ω, we have that $\Omega \subset \{ v_0 > 0 \}$ and $\mathcal{E}_p(v_0) \leq \mathcal{E}_p(v)$.

Proposition 4.7 (Existence)**.** *Let $n \geq 2$ and, if $n = 2$, assume that condition* (4.5) *holds true. Then there exists a solution to problem* (4.4).

Proof. Let $\{ u_k \}$ be a minimising sequence for problem (4.4). Without loss of generality, we may always assume that, for all $k \in \mathbb{N}$, $\mathcal{E}_p(u_k) \leq \mathcal{E}_p(0) = 0$, and, by Remark 4.6, $\Omega \subseteq \{ u_k > 0 \}$. Therefore we have

$$\begin{aligned} 0 \geq \mathcal{E}_p(u_k) &\geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \, dx + \beta \int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2 \right) d\mathcal{H}^{n-1} - 2 \int_\Omega f v \, dx \\ &\geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \, dx + \beta \int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2 \right) d\mathcal{H}^{n-1} - 2 \|f\|_{2,\Omega} \|u_k\|_{2,\Omega} \, , \end{aligned}$$

and by (4.7),

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \, dx + \beta \int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2 \right) d\mathcal{H}^{n-1} \leq C \|f\|_{2,\Omega} \, .$$

Then we have that there exists a positive constant still denoted by C, independent on the sequence $\{u_k\}$, such that

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \, dx + \int_{J_{u_k}} \left(\overline{u}_k^2 + \underline{u}_k^2 \right) d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \, dx < C. \tag{4.11}$$

The compactness theorem in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ (Theorem 2.14), ensures that there exists a subsequence $\{u_{k_j}\}$ and a function $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$, such that u_{k_j} converges to u strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$, weakly in $H^1(\Omega)$, almost everywhere in $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \leq \liminf_{j \to +\infty} \int_{\mathbb{R}^n} |\nabla u_{k_j}|^2 \, dx,$$

$$\int_{J_u} \left(\overline{u}^2 + \underline{u}^2 \right) d\mathcal{H}^{n-1} \leq \liminf_{j \to +\infty} \int_{J_{u_{k_j}}} \left(\overline{u}_{k_j}^2 + \underline{u}_{k_j}^2 \right) d\mathcal{H}^{n-1},$$

$$\mathcal{L}^n(\{ u > 0 \} \setminus \Omega) \leq \liminf_{j \to +\infty} \mathcal{L}^n(\{ u_{k_j} > 0 \} \setminus \Omega).$$

Finally we have

$$\mathcal{E}_p(u) \leq \liminf_{j \to +\infty} \mathcal{E}_p(u_{k_j}) = \inf \left\{ \mathcal{E}_p(v) \;\middle|\; v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega) \right\},$$

Therefore, u is a minimiser to problem (4.4). ☐

Theorem 4.8 (Euler-Lagrange equation). *Let u be a minimiser to problem (4.4), and let $v \in \mathrm{SBV}^{1/2}(\mathbb{R}^n)$ such that $J_v \subseteq J_u$. Assume that there exists $t > 0$ such that $\{ v > 0 \} \subseteq \{ u > t \}$ $\mathcal{L}^n$-a.e., and that*

$$\int_{J_u \setminus J_v} v^2 \, d\mathcal{H}^{n-1} < +\infty.$$

Then

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \beta \int_{J_u} \left(\overline{u} \gamma^+(v) + \underline{u} \gamma^-(v) \right) d\mathcal{H}^{n-1} = \int_\Omega f v \, dx, \tag{4.12}$$

where $\gamma^\pm = \gamma^\pm_{J_u}$.

Proof. Notice that since $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ with $J_v \subseteq J_u$ we have that $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$. Assume $v \in \mathrm{SBV}^{1/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. If $s \in \mathbb{R}$, recalling that $\{ v > 0 \} \subseteq \{ u > t \}$ $\mathcal{L}^n$-a.e.,

$$u(x) + sv(x) = u(x) \geq 0 \qquad \mathcal{L}^n\text{-a.e. } \forall x \in \{ u \leq t \},$$

while, for $|s|$ small enough,

$$u(x) + sv(x) \geq t - |s| \, \|v\|_\infty > 0 \qquad \forall x \in \{ u > t \}.$$

Therefore, we still have

$$u + sv \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{R}^+).$$

Moreover by minimality of u we have, for every $|s| \leq s_0$

$$\begin{aligned}
\mathcal{E}_p(u) &\leq \mathcal{E}_p(u + sv) \\
&= \int_{\mathbb{R}^n} |\nabla u + s \nabla v|^2 \, dx + \\
&\quad + \int_{J_{u+sv}} \left[\left(\gamma^+(u) + s\gamma^+(v) \right)^2 + \left(\gamma^-(u) + s\gamma^-(v) \right)^2 \right] d\mathcal{H}^{n-1} + \\
&\quad - 2 \int_{\mathbb{R}^n} f(u + sv) \, dx + C_0 \mathcal{L}^n(\{ u > 0 \}).
\end{aligned}$$

The set

$$S := \left\{ s \in [-s_0, s_0] \;\middle|\; \mathcal{H}^{n-1}(J_u \setminus J_{u+sv}) \neq 0 \right\}$$

is at most countable.

Let us define

$$\begin{aligned}
D_0 &= \left\{ x \in J_u \;\middle|\; \gamma^+(u)(x) \neq \gamma^-(u)(x) \right\}, \\
D_s &= \left\{ x \in J_u \;\middle|\; \gamma^+(u + sv)(x) \neq \gamma^-(u + sv)(x) \right\},
\end{aligned}$$

and notice that

$$\mathcal{H}^{n-1}(J_u \setminus D_0) = 0, \qquad \mathcal{H}^{n-1}(J_{u+sv} \setminus D_s) = 0.$$

Then we have to prove that

$$\left\{ s \in [-s_0, s_0] \;\middle|\; \mathcal{H}^{n-1}(D_0 \setminus D_s) \neq 0 \right\}$$

is at most countable. Observe that if $t \neq s$,

$$(D \setminus D_t) \cap (D \setminus D_s) = \emptyset.$$

Indeed if $x \in D \setminus D_s$

$$\gamma^+(u)(x) \neq \gamma^-(u)(x),$$
$$\gamma^+(u) + s\gamma^+(v)(x) = \gamma^-(u) + s\gamma^-(v)(x),$$

then

$$\gamma^+(v)(x) \neq \gamma^-(v)(x),$$

and so

$$s = \frac{\gamma^-(u)(x) - \gamma^+(u)(x)}{\gamma^+(v)(x) - \gamma^-(v)(x)}.$$

If $\mathcal{H}^0$ denotes the counting measure in $\mathbb{R}$, we can write

$$\int_{-s_0}^{s_0} \mathcal{H}^{n-1}(D_0 \setminus D_s)\, d\mathcal{H}^0 = \mathcal{H}^{n-1}\left(\bigcup_{(-s_0,\, s_0)} D_0 \setminus D_s \right) \leq \mathcal{H}^{n-1}(J_u) < +\infty,$$

then the claim is proved.

We are now able to differentiate in $s = 0$ the function $\mathcal{E}_p(u + sv)$, and observing that $0 \notin S$ is a minimum for $\mathcal{E}_p(u + sv)$, we get

$$\frac{1}{2}\dot{\mathcal{E}}_p(u)[v] = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \beta \int_{J_u} \left[\overline{u}\gamma^+(v) + \underline{u}\gamma^-(v)\right] d\mathcal{H}^{n-1} - \int_{\Omega} f v \, dx = 0.$$

If $v \notin L^\infty(\mathbb{R}^n)$, we consider $v_h = \min\{\, v, h \,\}$. Then

$$\dot{\mathcal{E}}_p(u)[v_h] = 0 \qquad \forall h > 0.$$

Observe that, since $\gamma^\pm(v_h) = \min\{\, \gamma^\pm(v), h \,\}$,

$$\gamma^\pm(v_h) \to \gamma^\pm(v) \qquad \mathcal{H}^{n-1}\text{-a.e. in } J_u.$$

Therefore, passing to the limit for $h \to +\infty$, by dominated convergence on the term

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v_h \, dx,$$

and by monotone convergence on the terms

$$\beta \int_{J_u} \left[\overline{u}\gamma^+(v_h) + \underline{u}\gamma^-(v_h)\right] d\mathcal{H}^{n-1}, \qquad \int_{\Omega} f v_h \, dx,$$

we get

$$0 = \lim_h \dot{\mathcal{E}}_p(u)[v_h] = \dot{\mathcal{E}}_p(u)[v].$$

□

We now want to use the Euler-Lagrange equation (4.12) to prove that if f belongs to $L^p(\Omega)$ with $p > n$, and if u is a minimiser to problem (4.4) then u belongs to $L^\infty(\mathbb{R}^n)$. In order to prove this, we need the following

Lemma 4.9. *Let m be a positive real number. There exists a positive constant $C = C(m, \beta, n)$ such that, for every function $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ with $\mathcal{L}^n(\{\, v > 0 \,\}) \leq m$,*

$$\left(\int_{\mathbb{R}^n} v^{2 \cdot 1^*} \, dx \right)^{\frac{1}{1^*}} \leq C \left[\int_{\mathbb{R}^n} |\nabla v|^2 \, dx + \beta \int_{J_v} \left(\overline{v}^2 + \underline{v}^2 \right) d\mathcal{H}^{n-1} \right],$$

where $1^* = \dfrac{n}{n-1}$ is the Sobolev conjugate of 1.

Proof. Classical Embedding of $BV(\mathbb{R}^n)$ in $L^{1^*}(\mathbb{R}^n)$ ensures that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,dx\right)^{\frac{1}{1^*}} \leq & C(n)\left|Dv^2\right|(\mathbb{R}^n)\\ =& C(n)\left[\int_{\mathbb{R}^n} 2v|\nabla v|\,dx + \int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)d\mathcal{H}^{n-1}\right].\end{aligned}$$

For every $\varepsilon > 0$, using Young's and Hölder's inequalities, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,dx\right)^{\frac{1}{1^*}} \leq & \frac{C(n)}{\varepsilon}\int_{\mathbb{R}^n} v^2\,dx +\\ & + C(n)\left[\varepsilon\int_{\mathbb{R}^n}|\nabla v|^2\,dx + \int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)d\mathcal{H}^{n-1}\right]\\ \leq & \frac{C(n)\,m^{\frac{1}{n}}}{\varepsilon}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,dx\right)^{\frac{1}{1^*}} +\\ & + C(n)\left[\varepsilon\int_{\mathbb{R}^n}|\nabla v|^2\,dx + \int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)d\mathcal{H}^{n-1}\right].\end{aligned}$$

Setting $\varepsilon = 2C(n)m^{\frac{1}{n}}$, we can find two constants $C(m,n), C(m,\beta,n) > 0$ such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\,dx\right)^{\frac{1}{1^*}} &\leq C(m,n)\left[\int_{\mathbb{R}^n}|\nabla v|^2\,dx + \int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)d\mathcal{H}^{n-1}\right]\\ &\leq C(m,\beta,n)\left[\int_{\mathbb{R}^n}|\nabla v|^2\,dx + \beta\int_{J_v}\left(\overline{v}^2+\underline{v}^2\right)d\mathcal{H}^{n-1}\right].\end{aligned}$$

□

We refer to [55] for the following lemma.

Lemma 4.10. *Let $g : [0,+\infty) \to [0,+\infty)$ be a decreasing function and assume that there exist $C, \alpha > 0$ and $\theta > 1$ constants such that for every $h > k \geq 0$,*

$$g(h) \leq C(h-k)^{-\alpha} g(k)^{\theta}.$$

Then there exists a constant $h_0 > 0$ such that

$$g(h) = 0 \qquad \forall h \geq h_0.$$

In particular, we have

$$h_0 = C^{\frac{1}{\alpha}} g(0)^{\frac{\theta-1}{\alpha}} 2^{\theta(\theta-1)}.$$

Proposition 4.11 (L^∞ bound)**.** *Let $n \geq 2$ and assume that, if $n = 2$, condition (4.5) holds true. Let $f \in L^p(\Omega)$, with $p > n$. Then there exists a constant $C = C(\Omega, f, p, \beta, C_0) > 0$ such that if u is a minimiser to problem (4.4), then*

$$\|u\|_\infty \leq C.$$

Proof. Let $\gamma^\pm = \gamma^\pm_{J_u}$. For every $\varphi, \psi \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ satisfying $J_\varphi, J_\psi \subseteq J_u$, define

$$a(\varphi,\psi) = \int_{\mathbb{R}^n}\nabla\varphi\cdot\nabla\psi\,dx + \beta\int_{J_u}\left[\gamma^+(\varphi)\gamma^+(\psi) + \gamma^-(\varphi)\gamma^-(\psi)\right]d\mathcal{H}^{n-1}.$$

For every v satisfying the assumptions of Theorem 4.8, it holds that

$$a(u, v) = \int_\Omega f v \, dx.$$

In particular, let us fix $k \in \mathbb{R}^+$ and define

$$\varphi_k(x) = \begin{cases} u(x) - k & \text{if } u(x) \geq k, \\ 0 & \text{if } u(x) < k, \end{cases}$$

then

$$\gamma^+(\varphi_k)(x) = \begin{cases} \overline{u}(x) - k & \text{if } \overline{u}(x) \geq k, \\ 0 & \text{if } \overline{u}(x) < k, \end{cases}$$

and analogously for $\gamma^-(\varphi_k)$. Furthermore, let us define

$$\mu(k) = \mathcal{L}^n(\{\, u > k \,\}).$$

We want to prove that $\mu(k) = 0$ for sufficiently large k. From Theorem 4.8, we have

$$a(u, \varphi_k) = \int_\Omega f \varphi_k \, dx, \tag{4.13}$$

and we can observe that

$$\begin{aligned} a(u, \varphi_k) &= \int_{\{\, u>k \,\}} |\nabla u|^2 \, dx + \beta \int_{J_u \cap \{\, u>k \,\}} [\overline{u}(\overline{u} - k) + \underline{u}(\underline{u} - k)] \, d\mathcal{H}^{n-1} \\ &\geq \int_{\{\, u>k \,\}} |\nabla u|^2 \, dx + \beta \int_{J_u \cap \{\, u>k \,\}} \left[(\overline{u} - k)^2 + (\underline{u} - k)^2\right] d\mathcal{H}^{n-1} \\ &= a(\varphi_k, \varphi_k). \end{aligned}$$

Moreover, by minimality, $\mathcal{E}_p(u) \leq \mathcal{E}_p(0) = 0$ and by Remark 4.6, $\Omega \subseteq \{\, u > 0 \,\}$. Therefore, (4.6) holds true and we can apply Lemma 4.9, having that there exists $C = C(\Omega, f, \beta, C_0) > 0$ such that

$$\int_\Omega f \varphi_k \, dx = a(u, \varphi_k) \geq a(\varphi_k, \varphi_k) \geq C \|\varphi_k\|^2_{2 \cdot 1^*}. \tag{4.14}$$

On the other hand

$$\begin{aligned} \int_\Omega f \varphi_k \, dx = \int_{\Omega \cap \{\, u>k \,\}} f(u - k) \, dx &\leq \left(\int_{\Omega \cap \{\, u>k \,\}} f^{\frac{2n}{n+1}} \, dx \right)^{\frac{n+1}{2n}} \|\varphi_k\|_{2 \cdot 1^*} \\ &\leq \|f\|_{p,\Omega} \, \|\varphi_k\|_{2 \cdot 1^*} \, \mu(k)^{\frac{n+1}{2n\sigma'}}, \end{aligned} \tag{4.15}$$

where

$$\sigma = \frac{p(n+1)}{2n} > 1,$$

since $p > n$. Joining (4.14) and (4.15), we have

$$\|\varphi_k\|_{2 \cdot 1^*} \leq C \|f\|_{p,\Omega} \, \mu(k)^{\frac{n+1}{2n\sigma'}}. \tag{4.16}$$

Let $h > k$, then

$$\begin{aligned} (h - k)^{2 \cdot 1^*} \mu(h) &= \int_{\{\, u>h \,\}} (h - k)^{2 \cdot 1^*} \, dx \\ &\leq \int_{\{\, u>h \,\}} (u - k)^{2 \cdot 1^*} \, dx \\ &\leq \int_{\{\, u>k \,\}} (u - k)^{2 \cdot 1^*} \, dx = \|\varphi_k\|^{2 \cdot 1^*}_{2 \cdot 1^*}. \end{aligned}$$

Using (4.16) and the previous inequality, we have

$$\mu(h) \leq C(h-k)^{-2\cdot 1^*}\mu(k)^{\frac{n+1}{(n-1)\sigma'}}.$$

Since $p > n$, then $\sigma' < (n+1)/(n-1)$. By Lemma 4.10, we have that $\mu(h) = 0$ for all $h \geq h_0$ with $h_0 = h_0(\Omega, f, \beta, C_0) > 0$, which implies

$$\|u\|_\infty \leq h_0.$$

□

Proof of Theorem 4.1. The result is obtained by joining Proposition 4.7 and Proposition 4.11. □

4.1.2 Density estimates for the jump set

In this subsection, we prove Theorem 4.2: in Proposition 4.17 we prove the lower bound for minimisers to problem (4.4); in Proposition 4.19 and Proposition 4.24 we prove the density estimates for the jump set of a minimiser to problem (4.4).

We will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $C^{1,1}$ boundary, and that $f \in L^p(\Omega)$, with $p > n$, is a positive function. To show that if u is a minimiser to problem (4.4) then u is bounded away from 0, we will first prove that there exists a positive constant δ such that $u > \delta$ almost everywhere in Ω, and then we will show that this implies the existence of a positive constant δ_0 such that $u > \delta_0$ almost everywhere in the set $\{\, u > 0 \,\}$. In the following, we will denote by $U_t := \{\, u < t \,\} \cap \Omega$.

Remark 4.12. Let u be a minimiser to (4.4), by Remark 4.6, u is a solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$$

Let $u_0 \in H^1_0(\Omega)$ be the solution to the following boundary value problem

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.17}$$

Then, by the maximum principle,

$$u \geq u_0 \quad \text{in } \Omega \subseteq \{\, u > 0 \,\} \quad \text{and} \quad \{\, u < t \,\} \cap \Omega = U_t \subseteq \{\, u_0 < t \,\} \cap \Omega.$$

Lemma 4.13. *There exist two positive constants $t_0 = t_0(\Omega, f)$ and $C = C(\Omega, f)$ such that if u is a minimiser to* (4.4) *then for every $t \in [0, t_0]$ it results*

$$\mathcal{L}^n(U_t) \leq C\, t. \tag{4.18}$$

Proof. Let u_0 be the solution to (4.17), fix $\varepsilon > 0$ such that the set

$$\Omega_\varepsilon = \{\, x \in \Omega \mid d(x, \partial\Omega) > \varepsilon \,\}$$

is not empty. Since u_0 is superharmonic and non-negative in Ω, by maximum principle we have that

$$\alpha = \inf_{\Omega_\varepsilon} u_0 > 0.$$

then u_0 solves

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \\ u_0 \geq \alpha & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Therefore, if we consider the solution v to the following boundary value problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \setminus \bar{\Omega}_\varepsilon, \\ v = 0 & \text{on } \partial\Omega, \\ v = \alpha & \text{in } \bar{\Omega}_\varepsilon, \end{cases}$$

we have that $u \geq u_0 \geq v$ almost everywhere in Ω and

$$\{ u < t \} \cap \Omega = U_t \subseteq \{ u_0 < t \} \cap \Omega \subseteq \{ v < t \} \cap \Omega.$$

The Hopf Lemma implies that
there exists a constant $\tau = \tau(\Omega) > 0$ such that

$$\frac{\partial v}{\partial \nu} < -\tau \qquad \text{on } \partial\Omega.$$

Let $x \in \bar{\Omega}$, and let x_0 be a projection of x onto the boundary $\partial\Omega$, then

$$|x - x_0| = d(x, \partial\Omega), \qquad \frac{x - x_0}{|x - x_0|} = -\nu_\Omega(x_0),$$

where ν_Ω denotes the exterior normal to $\partial\Omega$. We can write

$$\begin{aligned} v(x) &= \underbrace{v(x_0)}_{=0} + \nabla v(x_0) \cdot (x - x_0) + o(|x - x_0|) \\ &= -\frac{\partial v}{\partial \nu}(x_0)|x - x_0| + o(|x - x_0|) \\ &\geq \tau |x - x_0| + o(|x - x_0|) \\ &> \frac{\tau}{2}|x - x_0| = \frac{\tau}{2} d(x, \partial\Omega) \end{aligned} \tag{4.19}$$

for every x such that $d(x, \partial\Omega) < \sigma_0$ for a suitable $\sigma_0 = \sigma_0(\Omega, f) > 0$. Notice that if $\bar{x} \in \overline{\Omega}$ and $\lim_{x \to \bar{x}} v(x) = 0$ then necessarily $\bar{x} \in \partial\Omega$. Therefore, there exists a $t_0 = t_0(\Omega, f) > 0$ such that $v(x) < t_0$ implies $d(x, \partial\Omega) < \sigma_0$. Consequently, if $t < t_0$, we have that

$$\{ v < t \} \subseteq \{ d(x, \partial\Omega) < \sigma_0 \},$$

and by (4.19), we get

$$\mathcal{L}^n(U_t) \leq \mathcal{L}^n(\{ v < t \}) \leq \mathcal{L}^n\left(\left\{ x \in \Omega \;\middle|\; d(x, \partial\Omega) \leq \frac{2}{\tau} t \right\} \right) \leq C(\Omega) \frac{2}{\tau} t.$$

if t is small enough, since Ω is $C^{1,1}$.

□

Lemma 4.14. *Let $g : [0, t_1] \to [0, +\infty)$ be an increasing, absolutely continuous function such that*

$$g(t) \leq C t^\alpha \big(g'(t)\big)^\sigma \qquad \forall t \in [0, t_1], \tag{4.20}$$

with $C > 0$ and $\alpha > \sigma > 1$. Then there exists $t_0 > 0$ such that

$$g(t) = 0 \qquad \forall t \leq t_0.$$

Precisely,

$$t_0 = \left(\frac{C(\alpha - \sigma)}{\sigma - 1} g(t_1)^{\frac{\sigma - 1}{\sigma}} + t_1^{\frac{\sigma - \alpha}{\sigma}} \right)^{\frac{\sigma}{\sigma - \alpha}}.$$

Proof. Assume by contradiction that $g(t) > 0$ for every $t > 0$. Inequality (4.20) implies

$$\frac{g'}{g^{\frac{1}{\sigma}}} \geq \frac{1}{C} t^{-\frac{\alpha}{\sigma}}.$$

Integrating between t_0 and t_1, we have

$$\frac{\sigma}{\sigma-1}\left(g(t_1)^{\frac{\sigma-1}{\sigma}} - g(t_0)^{\frac{\sigma-1}{\sigma}}\right) \geq \frac{1}{C}\frac{\sigma}{\sigma-\alpha}\left(t_1^{\frac{\sigma-\alpha}{\sigma}} - t_0^{\frac{\sigma-\alpha}{\sigma}}\right).$$

Since $\alpha > \sigma > 1$, we have

$$0 \leq g(t_0)^{\frac{\sigma-1}{\sigma}} \leq \frac{\sigma-1}{C(\alpha-\sigma)}\left(t_1^{\frac{\sigma-\alpha}{\sigma}} - t_0^{\frac{\sigma-\alpha}{\sigma}}\right) + g(t_1)^{\frac{\sigma-1}{\sigma}},$$

which is a contradiction if

$$t_0 \leq \left(\frac{C(\alpha-\sigma)}{\sigma-1} g(t_1)^{\frac{\sigma-1}{\sigma}} + t_1^{\frac{\sigma-\alpha}{\sigma}}\right)^{\frac{\sigma}{\sigma-\alpha}}.$$

□

Remark 4.15. Let g be as in Lemma 4.14 and assume that $g(t_1) \leq K$, then $g(t) = 0$ for all $0 < t < \tilde{t}$ where

$$\tilde{t} = \left(\frac{C(\alpha-\sigma)}{\sigma-1} K^{\frac{\sigma-1}{\sigma}} + t_1^{\frac{\sigma-\alpha}{\sigma}}\right)^{\frac{\sigma}{\sigma-\alpha}}.$$

We now have the tools to prove the lower bound inside Ω.

Proposition 4.16. *There exists a positive constant $\delta = \delta(\Omega, f, p, \beta, C_0) > 0$ such that if u is a minimiser to problem (4.4) then*

$$u \geq \delta$$

almost everywhere in Ω.

Proof. Assume that Ω is connected and define the function

$$u_t(x) = \begin{cases} \max\{\, u, t \,\} & \text{in } \Omega, \\ u & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Recalling that $U_t = \{\, u < t \,\} \cap \Omega$, we have

$$J_{u_t} \setminus \partial^* U_t = J_u \setminus \partial^* U_t,$$

and on this set $\underline{u_t} = \underline{u}$ and $\overline{u_t} = \overline{u}$.

Then we get by minimality of u, and using the fact that $J_{u_t} \cap \partial^* U_t \subseteq \partial\Omega$,

$$\begin{aligned}
0 &\geq \mathcal{E}_p(u) - \mathcal{E}_p(u_t) \\
&= \int_{U_t} |\nabla u|^2 \, dx - 2\int_{U_t} f(u-t)\, dx + \beta \int_{\partial^* U_t \cap J_u} \left(\underline{u}^2 + \overline{u}^2\right) d\mathcal{H}^{n-1} + \\
&\quad - \beta \int_{J_{u_t} \cap \partial^* U_t \cap J_u} \left[t^2 + (\gamma^+_{\partial\Omega}(u))^2\right] d\mathcal{H}^{n-1} - \beta \int_{\left(J_{u_t} \cap \partial^* U_t\right) \setminus J_u} \left(t^2 + u^2\right) d\mathcal{H}^{n-1} \\
&\geq \int_{U_t} |\nabla u|^2 \, dx - 2\beta t^2 \mathcal{H}^{n-1}(\partial^* U_t \cap \partial\Omega)
\end{aligned}$$

where we ignored all the non-negative terms except the integral of $|\nabla u|^2$, and we used that $u \le t$ in $\partial^* U_t \setminus J_u$. By Lemma 4.13, we can choose t small enough to have $\mathcal{L}^n(U_t) \le \mathcal{L}^n(\Omega)/2$, then applying the isoperimetric inequality in Theorem 2.6 to the set $E = U_t$, we get

$$\int_{U_t} |\nabla u|^2 \, dx \le 2\beta \, C \, t^2 \, P(U_t; \Omega). \tag{4.21}$$

Let us define

$$p(t) = P(U_t; \Omega),$$

and consider the absolutely continuous function

$$g(t) = \int_{U_t} u \, |\nabla u| \, dx = \int_0^t s \, p(s) \, ds.$$

By minimality of u we can apply the a priori estimates (4.11) to prove the equiboundedness of g, i.e. there exists $K = K(\Omega, f, \beta, C_0) > 0$ such that $g(t) \le K$ for all $t > 0$. Using the Hölder inequality and the estimate (4.21), we have

$$g(t) \le \left(\int_{U_t} u^2 \, dx \right)^{\frac{1}{2}} \left(\int_{U_t} |\nabla u|^2 \, dx \right)^{\frac{1}{2}} \le \sqrt{2\beta C} \, t \, \mathcal{L}^n(U_t)^{\frac{1}{2}} (t^2 p(t))^{\frac{1}{2}}.$$

Fix $1 > \varepsilon > 0$. Then we can write $\mathcal{L}^n(U_t) = \mathcal{L}^n(U_t)^{\varepsilon} \mathcal{L}^n(U_t)^{1-\varepsilon}$, and by Lemma 4.13 there exists a constant $C = C(\Omega, f, \beta) > 0$ such that

$$g(t) \le C \, t^{2 + \frac{1-\varepsilon}{2}} \, \mathcal{L}^n(U_t)^{\frac{\varepsilon}{2}} \, p(t)^{\frac{1}{2}}.$$

By the relative isoperimetric inequality in Theorem 2.5, we can estimate

$$\mathcal{L}^n(U_t)^{\frac{\varepsilon}{2}} \le C(\Omega, n) p(t)^{\frac{\varepsilon n}{2(n-1)}},$$

and, noticing that $p(t) = g'(t)/t$, we get

$$g(t) \le C t^{\alpha} (g'(t))^{\sigma},$$

where

$$\alpha = 2 - \frac{\varepsilon}{2}\left(1 + \frac{n}{n-1}\right), \qquad \sigma = \frac{1}{2} + \frac{\varepsilon}{2} \frac{n}{n-1}.$$

In particular, if we choose

$$\varepsilon \in \left(\frac{n-1}{n}, \frac{3n-3}{3n-1} \right),$$

we have that $\alpha > \sigma > 1$, and then, using Lemma 4.14 and Remark 4.15, there exists a $\delta = \delta(\Omega, f, p, \beta, C_0) > 0$ such that $g(t) = 0$ for every $t < \delta$. Then $\mathcal{L}^n(\{ u < t \} \cap \Omega) = 0$ for every $t < \delta$, hence

$$u \ge \delta$$

almost everywhere in Ω.

When Ω is not connected, then

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_N,$$

with Ω_i pairwise disjoint connected open sets. Using u_t as the function u truncated inside a single Ω_i, we find constants $\delta_i > 0$ such that

$$u(x) \ge \delta_i$$

almost everywhere in Ω_i. Therefore choosing $\delta = \min \{ \delta_1, \ldots, \delta_N \}$ we have $u(x) > \delta$ almost everywhere in Ω. □

Using the previous result, we have the following proposition.

Proposition 4.17 (Lower Bound). *There exists a positive constant $\delta_0 = \delta_0(\Omega, f, p, \beta, C_0)$ such that if u is a minimiser to problem (4.4) then*

$$u \geq \delta_0$$

almost everywhere in $\{ u > 0 \}$.

Proof. We observe that, by Proposition 4.16, $\Omega \subseteq \{ u > t \}$, so that the function $u\chi_{\{u>t\}}$ is an admissible competitor for u. Then we can follow the proof of Theorem 3.5 to prove the assertion. □

Remark 4.18. From Proposition 4.17, if u is a minimiser to problem (4.4), we have that

$$\partial^* \{ u > 0 \} \subseteq J_u \subseteq K_u. \tag{4.22}$$

Indeed, on $\partial^* \{ u > 0 \}$ we have that, by definition, $\underline{u} = 0$ and that, since $u \geq \delta_0$ $\mathcal{L}^n$-a.e. in $\{ u > 0 \}$, $\overline{u} \geq \delta_0$.

Proposition 4.19 (Density Estimates). *There exist positive constants $C = C(\Omega, f, p, \beta, C_0)$, $c = c(\Omega, f, p, \beta, C_0)$ and $\delta_1 = \delta_1(\Omega, f, p, \beta, C_0)$ such that if u is a minimiser to problem (4.4) then for every $B_r(x)$ such that $B_r(x) \cap \Omega = \emptyset$, we have:*

(a) For every $x \in \mathbb{R}^n \setminus \Omega$,

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq Cr^{n-1}; \tag{4.23}$$

(b) For every $x \in K_u$,

$$\mathcal{L}^n(B_r(x) \cap \{ u > 0 \}) \geq cr^n; \tag{4.24}$$

(c) The function u has bounded support, namely

$$\{ u > 0 \} \subseteq B_{1/\delta_1}.$$

Proof. This is a consequence of the lower bound in Proposition 4.17 and the upper bound in Proposition 4.11. In particular, the proof of Points *(a)* and *(b)* follows exactly as the one of equations (3.11) and (3.12) in Proposition 3.7. To prove *(c)*, let $\delta_1 > 0$ and $x \in K_u$ such that $d(x, \partial\Omega) \geq 1/\delta_1$. From (4.24), noticing that $\mathcal{E}_p(u) \leq \mathcal{E}_p(0) = 0$, we have that

$$c\,\delta_1^{-n} \leq \mathcal{L}^n(\{ u > 0 \} \setminus \Omega) \leq \frac{2\|u\|_\infty}{C_0} \int_\Omega f\,dx,$$

which is a contradiction if δ_1 is sufficiently small. The assertion then follows using (4.22). □

Remark 4.20. Given the summability assumption on the function f and the lower bound given in Proposition 4.17, we have that minimisers to (4.4) are almost-quasi-minimisers of the functional $\mathcal{M}$, defined on $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$ as

$$\mathcal{M}(v) = \int_{\mathbb{R}^n} |\nabla v|^2\,dx + \lambda\mathcal{H}^{n-1}(J_v),$$

that is, there exists $C(\Omega, f, p, \beta, C_0) > 0$, $\Lambda(\Omega, f, p, \beta, C_0) \geq \lambda$ and $\alpha(n, p) > n-1$ such that, if $B_r(x)$ is a ball of radius $r \leq 1$, and $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$, with $\{ u \neq v \} \subset B_r(x)$, then

$$\mathcal{M}_\lambda(u; B_r(x)) \leq \mathcal{M}_\Lambda(v; B_r(x)) + Cr^\alpha,$$

where
$$\mathcal{M}_\lambda(v; B_r(x)) := \int_{B_r(x)} |\nabla v|^2\, dx + \lambda \mathcal{H}^{n-1}(J_v \cap B_r(x)).$$

Indeed, let u be a minimiser to (4.4), let $B_r(x)$ be a ball of radius $r \leq 1$, and let $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap H^1(\Omega)$, with $\{ u \neq v \} \subset B_r(x)$, and let

$$w = \min \{ \max \{ v, 0 \}, \|u\|_\infty \}.$$

By minimality of u we have that

$$\mathcal{M}_\lambda(u; B_r(x)) \leq \mathcal{M}_\Lambda(v; B_r(x)) + Cr^n,$$

where $\lambda = \beta\delta_0^2$ and $\Lambda = 2\beta\|u\|_\infty^2$. Moreover,

$$\int_{\Omega\cap B_r(x)} fu\, dx \leq \|f\|_{p,\Omega}\|u\|_\infty \mathcal{L}^n(B_r)^{1/p'} = C(\Omega, f, p, \beta, C_0)r^\alpha,$$

where
$$n > \alpha = \frac{n}{p'} > n - 1.$$

Finally, we have
$$\mathcal{M}_\lambda(u; B_r(x)) \leq \mathcal{M}_\Lambda(v; B_r(x)) + Cr^\alpha.$$

Such a minimality property can be used to prove the lower density estimate in Proposition 4.24 using the following decay lemma

Lemma 4.21 (Decay lemma)**.** *Let $1 > \gamma > n - \alpha$. There exists $\tau_0 = \tau_0(n, \Omega, \gamma, \lambda) > 0$ such that for every $\tau_0 > \tau > 0$ there exist $r_0 = r_0(\tau, \Omega)$, $\varepsilon_0 = \varepsilon_0(\tau, \Omega) > 0$ such that, if $x_0 \in \partial\Omega$, $r_0 > r > 0$, and u is a almost-quasi minimiser on $B_r = B_r(x_0)$ for the functional $\mathcal{G}$ such that*

$$\mathcal{H}^{n-1}(J_u \cap B_r) \leq \varepsilon_0 r^{n-1},$$

then we have that either
$$\mathcal{M}_\lambda(u; B_r) \leq r^{n-\gamma},$$

or
$$\mathcal{M}_\lambda(u; B_{\tau r}) \leq \tau^{n-\gamma}\mathcal{M}_\lambda(u; B_r).$$

Proof. The proof of the decay lemma is similar to the one of Lemma 3.12 (see also [28, Lemma 5.3], [22, Section 4], [44, Lemma 4.9]). The main difference is in the construction of the blow-up sequence of almost-quasi minimisers.

Let u_k be a sequence of almost-quasi minimisers on B_{r_k} contradicting the lemma, with $\lim_k r_k = 0$. To reach a contradiction, one usually constructs a sequence of functions $\tilde{v}_k$ on the unit ball, related to the sequence u_k, that converges to a harmonic function v. Then, we construct a sequence of admissible test functions ψ_k on B_{r_k} and use the minimality property of u_k to prove that v is harmonic. If $d(x_0, \Omega) > 0$, then the test function are only required to be in $\mathrm{SBV}(B_{r_k})$, while, if $x_0 \in \partial\Omega$ the additional constraint $\psi_k \in \mathrm{SBV}(B_{r_k}) \cap H^1(\Omega \cap B_{r_k})$ should be treated with more carefulness.

Without loss of generality let $x_0 = 0$ and let $E_k = r_k^{2-n}\mathcal{M}_\lambda(u_k; B_{r_k})$, and define

$$v_k(x) = \frac{1}{E_k^{1/2}} u_k(r_k x).$$

For any k, we extend $u_k \in H^1(\Omega \cap B_{r_k})$ to $Lu_k \in H^1(B_{r_k})$, which is a function such that $u_k - Lu_k \equiv 0$ in Ω. Let us define, with a slight abuse of notation,

$$Lv_k(x) = \frac{1}{E_k^{1/2}} Lu_k(r_k x), \qquad w_k = v_k - Lv_k,$$

so that, by construction, and by properties of the blow-up,

$$\liminf_k \mathcal{L}^n(\{w_k = 0\}) \geq \liminf_k r_k^{-n} \mathcal{L}^n(\Omega \cap B_{r_k}) > 0. \tag{4.25}$$

This is the key property: by Poincaré inequality in SBV, there exist $\tilde{w}_k$ truncated functions, such that

$$\lim_k \mathcal{L}^n(\{w_k \neq \tilde{w}_k\}) = 0, \tag{4.26}$$

and, up to subtracting medians, w_k converge in L^2 to some Sobolev function. By (4.25) and (4.26), and considering that $\tilde{w}_k$ is a truncation of w_k, then for big enough k, up to $\mathcal{L}^n$-negligible sets,

$$\{w_k = 0\} \subseteq \{w_k = \tilde{w}_k\}.$$

This means that if we define $\tilde{v}_k = \tilde{w}_k + Lv_k$, then the scaled back functions

$$\tilde{u}_k(x) := E_k^{1/2} \tilde{v}_k\left(\frac{x}{r_k}\right)$$

respect the property $\tilde{u}_k \equiv u_k$ in $\Omega \cap B_{r_k}$. Moreover, it is possible to choose an extension L (see Lemma 4.22) such that, combining the Poincaré inequality in SBV and the Poincaré inequality in H^1, then there exist constants c_k such that $\tilde{v}_k - c_k$ converge in L^2 to a function $v \in H^1(B_1)$. This ensures that, if we take $\rho < \rho'$ small enough, η cut-off functions between B_ρ and $B_{\rho'}$, and $\varphi \in H^1(B_1)$, then the test functions $\psi_k = E_k^{1/2}\varphi_k(x/r_k)$, with

$$\varphi_k = \Big(\eta\big(\varphi + c_k\big) + (1-\eta)\tilde{v}_k\Big)\chi_{B_{\rho'}} + v_k \chi_{B_1 \setminus B_{\rho'}},$$

are admissible test functions for any $\varphi \in H^1(B_1)$, leading to similar computations as Lemma 3.12 (and the aforementioned papers). $\square$

Lemma 4.22. *Let Ω be an open set with Lipschitz boundary, and let $x_0 \in \partial\Omega$. There exist positive constants $\rho_0 = \rho_0(\Omega, x_0)$, $C = C(\Omega, x_0)$, $\delta = \delta(\Omega, x_0) > 1$, and an extension operator*

$$L : H^1(\Omega) \to H^1(B_{\rho_0}(x_0))$$

such that, for any $u \in H^1(\Omega)$, and for any $r < \rho_0$, we have that $Lu \equiv u$ in $\Omega \cap B_{\rho_0}(x_0)$ and

$$\int_{B_r(x_0)} |\nabla Lu|^2 \, dx \leq C \int_{\Omega \cap B_{\delta r}(x_0)} |\nabla u|^2 \, dx. \tag{4.27}$$

Proof. We can assume without loss of generality that $x_0 = 0$, and, if s is small enough, we have that, up to rotations,

$$\Omega \cap B_s = \left\{ \, (x', x_n) \in B_s \;\middle|\; \gamma(x') < x_n \, \right\},$$

for a suitable Lipschitz function γ, with $\gamma(0) = 0$. We denote by Φ the diffeomorphism that flattens the boundary $\partial\Omega$, namely

$$\Phi(x', x_n) = (x', x_n - \gamma(x')), \qquad \Phi^{-1}(y', y_n) = (y', y_n + \gamma(y')).$$

Let $M = \|\nabla\gamma\|_\infty$, we claim that for any $r < (1+M)^{-2}s$ we have

$$\Phi(B_r) \subset B_{(1+M)r} \subset \Phi(B_{(1+M)^2 r}). \tag{4.28}$$

Indeed, let $x \in B_r$, then

$$|\Phi(x)|^2 \leq |x|^2 + 2|x_n\gamma(x)| + |\gamma(x)|^2,$$

so that, we have

$$|\gamma(x)| \leq |x|\|\nabla\gamma\|_\infty,$$

and then

$$|\Phi(x)| \leq (1+M)r.$$

In a similar way, we have that for any $x \in B_{(1+M)r}$,

$$|\Phi^{-1}(x)| \leq (1+M)^2 r,$$

thus the claim is proved.

Let us take a ball B_t such that $\Phi^{-1}(B_t) \subset B_s$, which we can find thanks to (4.28), and let us reflect the function $v(x) = u(\Phi(x))$ as follows: for any $x \in B_t$, we define

$$Lv(x) = \begin{cases} v(x) & \text{if } x_n < 0, \\ -3v(x', -x_n) + 4v\big(x', -\frac{x_n}{2}\big) & \text{if } x_n > 0, \end{cases}$$

which is still a Sobolev function in B_t. Moreover, we have

$$\int_{B_t} |\nabla Lv|^2\, dx \leq C \int_{B_t \cap \{x_n < 0\}} |\nabla v|^2\, dx,$$

where C is independent of Ω. We put $Lu(x) = Lv(\Phi^{-1}(x))$, and by change of variables, we get

$$\int_{\Phi^{-1}(B_t)} |\nabla Lu|^2\, dx \leq C(\Omega) \int_{\Omega \cap \Phi^{-1}(B_t)} |\nabla u|^2\, dx. \tag{4.29}$$

Finally, taking $\rho_0 = (1+M)^{-2}s$, and $t_0 = (1+M)^{-1}s$, we have $B_{\rho_0} \subset \Phi^{-1}(B_{t_0})$. Therefore, denoting by $\delta = (1+M)^2$, by (4.28) and (4.29), we get, for $r < \rho_0$,

$$\int_{B_r(x_0)} |\nabla Lu|^2\, dx \leq \int_{\Phi^{-1}(B_{\sqrt{\delta}r})} |\nabla Lu|^2\, dx \leq C \int_{\Omega \cap \Phi^{-1}(B_{\sqrt{\delta}r})} |\nabla u|^2\, dx \leq C \int_{\Omega \cap B_{\delta r}} |\nabla u|^2\, dx.$$

□

Remark 4.23. Notice that if Ω is bounded, the constants in Lemma 4.22 can be chosen independent of the point x_0.

Then, as in the proof of Theorem 3.2, arguing by contradiction, we can iterate the decay lemma (Lemma 4.21) to prove the following lower density estimates.

Proposition 4.24 (Lower Density Estimate)**.** *There exists a positive constant $c = c(\Omega, f, p, \beta, C_0)$ such that if u is a minimiser to problem* (4.4) *then*

1. *For any $x \in K_u$*
$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \geq cr^{n-1};$$
2. *J_u is essentially closed, namely*
$$\mathcal{H}^{n-1}(K_u \setminus J_u) = 0;$$

Finally, we have

Proof of Theorem 4.2. The result is obtained by joining Proposition 4.17, Proposition 4.19, and Proposition 4.24. □

Remark 4.25. Let u be a minimiser to (4.4) and let $A = \{\, \overline{u} > 0 \,\} \setminus K_u$, then arguing as in Remark 3.14 and using the fact that u is superharmonic in the complement of k_u, we have that A is open and $\partial A = K_u$. In particular, the pair (A, u) is then a minimiser for the functional

$$\mathcal{E}_p(E, v) = \int_E |\nabla v|^2 \, dx - 2 \int_\Omega f v \, dx + \int_{\partial E} \left(\underline{v}^2 + \overline{v}^2 \right) d\mathcal{H}^{n-1} + C_0 \mathcal{L}^n(E \setminus \Omega)$$

over all pairs (E, v) with E open set of finite perimeter containing Ω and $v \in H^1(E)$.

4.2 Thin insulating layer: the limit problem

The main results of this section are based on the results of [36]. We address the problem of qualitatively and quantitatively describing the shape of the optimal configuration of insulating material in the thin layer setting.

More precisely, as in Section 3.2, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded, open set, and let $h\colon \partial\Omega \to \mathbb{R}$ be a positive smooth function. Denoting by ν_0 the exterior unit normal to the boundary of Ω, for every $\varepsilon > 0$ sufficiently small, we define

$$\Sigma_\varepsilon = \{\, \sigma + t\nu_0(\sigma) \mid \sigma \in \partial\Omega,\ 0 < t < \varepsilon h(\sigma) \,\}$$

and we denote by $\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon$. Moreover, we assume $k = \varepsilon$.

We then consider the minimisation of the energy

$$\mathcal{E}_{p,\varepsilon}(v, h) = \int_\Omega |\nabla v|^2 \, dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} v^2 \, d\mathcal{H}^{n-1} - 2 \int_\Omega f v \, dx, \tag{4.30}$$

where $v \in H^1(\Omega_\varepsilon)$. Similar problems have been studied before with a Dirichlet boundary condition in [16], [6], [18], and more recently in [10] (see also [9]), where the assumptions on the regularity of Ω have been weakened.

As for the capacitary problem, instead of penalising the volume of the insulator we displace, we will work under a volume constraint assumption. Namely, for a given $m > 0$, the problem of finding the best configuration of insulating material surrounding Ω, in the thin layer setting, can be stated as

$$\min \left\{ \mathcal{E}_{p,\varepsilon}(v, h) \;\middle|\; \begin{array}{l} v \in H^1(\Omega_\varepsilon), \\ \mathcal{L}^n(\Sigma_\varepsilon) \le \varepsilon m \end{array} \right\}. \tag{4.31}$$

Recall that in the limit, the volume constraint can be expressed by an integral constraint on h, namely, working in the space

$$\mathcal{H}_m = \mathcal{H}_m(\partial\Omega) = \left\{ h \in L^1(\partial\Omega) \;\middle|\; \begin{array}{l} \displaystyle\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} \le m \\ h \ge 0 \end{array} \right\}.$$

In subsection 4.2.1 we compute the Γ-limit, in the strong $L^2(\mathbb{R}^n)$ topology of the functional $\mathcal{E}_{p,\varepsilon}(\cdot, h)$. Then in subsection 4.2.2 we study the minimum of the limit energy in $H^1(\Omega) \times \mathcal{H}_m$. We remark that no assumption on the sign of the function f is needed.

4.2.1 The limit problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive Lipschitz function $h\colon \partial\Omega \to \mathbb{R}$. In the notations of Section 2.2 let

$$\Gamma_t = \{\, x \in \mathbb{R}^n \mid d(x) < t \,\} \setminus \Omega.$$

From the regularity of $\partial\Omega$, there exists $d_0 > 0$ such that for every $x \in \Gamma_{d_0}$, we can uniquely write

$$x = \sigma(x) + d(x)\nu_0(\sigma(x)),$$

where $\sigma(x)$ is the metric projection of x on Ω and $d(x)$ is its distance from Ω. Moreover, on Γ_{d_0}, extending ν_0 as $\nu_0(x) = \nu_0(\sigma(x))$ we have that ν_0 is orthogonal to the level set of the distance. In the following we will assume h to be extended to Γ_{d_0} as $h(x) = h(\sigma(x))$ and $\varepsilon > 0$ such that $\varepsilon\|h\|_\infty < d_0$, so that $\Sigma_\varepsilon \subset \Gamma_{d_0}$.

We extend the energy to the whole $L^2(\mathbb{R}^n)$

$$\mathcal{E}_{p,\varepsilon}(v,h) = \begin{cases} \displaystyle\int_\Omega |\nabla v|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon}|\nabla v|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx & \text{if } v \in H^1(\Omega_\varepsilon),\\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus H^1(\Omega_\varepsilon),\end{cases}$$

and define the energy

$$\mathcal{E}_{p,0}(v,h) = \begin{cases} \displaystyle\int_\Omega |\nabla v|^2\,dx + \beta\int_{\partial\Omega} \frac{v^2}{1+\beta h}\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx & \text{if } v \in H^1(\Omega),\\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus H^1(\Omega).\end{cases}$$

In [36] they prove the following theorem

Theorem 4.26. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a Lipchitz function $h\colon \partial\Omega \to (0,+\infty)$. Then $\mathcal{E}_{p,\varepsilon}(\cdot,h)$ Γ-converges, as $\varepsilon \to 0^+$, in the strong $L^2(\mathbb{R}^n)$ topology, to $\mathcal{E}_{p,0}(\cdot,h)$.*

Proof. As previously stated, the proof of this theorem is analogous to the one of Proposition 3.17; hence, in the following, we will only remark the main differences between the two.

We start by proving the Γ-liminf inequality: Let $v \in L^2(\mathbb{R}^n)$ and let $v_\varepsilon \in L^2(\mathbb{R}^n)$ such that v_ε converges to v in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Up to passing to a sub-sequence, we can assume that

$$\liminf_{\varepsilon\to 0^+} \mathcal{E}_{p,\varepsilon}(v_\varepsilon,h) = \lim_{\varepsilon\to 0^+} \mathcal{E}_{p,\varepsilon}(v_\varepsilon,h),$$

moreover, we can assume that such a limit is finite and that $v_\varepsilon \in H^1(\Omega_\varepsilon)$. Therefore, there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla v_\varepsilon|^2\,dx - 2\int_\Omega fv_\varepsilon\,dx \le \mathcal{E}_{p,\varepsilon}(v_\varepsilon,h) < C. \tag{4.32}$$

Since v_ε converges in $L^2(\mathbb{R}^n)$ to v, we have that

$$\int_\Omega fv_\varepsilon\,dx \to \int_\Omega fv\,dx$$

which, together with (4.32), implies the boundedness in $H^1(\Omega)$ of v_ε. Then $v \in H^1(\Omega)$ and, up to a subsequence, v_ε converges to v, weakly in $H^1(\Omega)$. As a consequence

$$\liminf_{\varepsilon\to 0^+} \int_\Omega |\nabla v_\varepsilon|^2\,dx - 2\int_\Omega fv_\varepsilon\,dx \ge \int_\Omega |\nabla v|^2\,dx - 2\int_\Omega fv\,dx.$$

We are left to show that

$$\liminf_{\varepsilon\to0^+}\varepsilon\int_{\Sigma_\varepsilon}|\nabla v_\varepsilon|^2\,dx+\beta\int_{\Omega_\varepsilon}v_\varepsilon^2\,d\mathcal{H}^{n-1}\geq\beta\int_{\partial\Omega}\frac{v^2}{1+\beta h}\,d\mathcal{H}^{n-1}. \tag{4.33}$$

Using (3.43), (3.44) we have that

$$\int_{\Sigma_\varepsilon}|\nabla v_\varepsilon|^2\,dx\geq\int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)}|\nabla v_\varepsilon(\sigma+t\nu_0)|^2(1-\varepsilon Q_0)\,dt\,d\mathcal{H}^{n-1}$$

and

$$\int_{\partial\Omega_\varepsilon}v_\varepsilon^2\,d\mathcal{H}^{n-1}\geq\int_{\partial\Omega}v_\varepsilon^2(\sigma+\varepsilon h(\sigma)\nu_0)(1-\varepsilon Q_0)\,d\mathcal{H}^{n-1}$$

for some constant $Q_0>0$. Following the same computations as the proof of Proposition 3.17, one can easily prove that

$$\varepsilon\int_{\Sigma_\varepsilon}|\nabla v_\varepsilon|^2\,dx+\beta\int_{\Omega_\varepsilon}v_\varepsilon^2\,d\mathcal{H}^{n-1}\geq\beta\int_{\partial\Omega}\frac{v_\varepsilon^2}{1+\beta h}\,d\mathcal{H}^{n-1}-\varepsilon Q_0R_\varepsilon(\varepsilon,v_\varepsilon),$$

where, by (3.45) and (3.46),

$$\begin{aligned}R_\varepsilon(\varepsilon,v_\varepsilon)&\leq C\left(|\nabla v_\varepsilon|^2\,dx+\beta\int_{\Omega_\varepsilon}v_\varepsilon^2\,d\mathcal{H}^{n-1}\right)\\&\leq\mathcal{E}_{p,\varepsilon}(h,v_\varepsilon)+2\int_\Omega fv_\varepsilon\,dx<C.\end{aligned}$$

Then, by the weak convergence in $H^1(\Omega)$ of v_ε to v, and by the lower semicontinuity in $L^2(\partial\Omega)$ of the function

$$v\mapsto\beta\int_{\partial\Omega}\frac{v^2}{1+\beta h}\,d\mathcal{H}^{n-1},$$

we have (4.33) and the Γ-liminf inequality.

Γ-limsup inequality: Let $v\in L^2(\mathbb{R}^n)$, if $v\notin H^1(\Omega)$ the Γ-limsup inequality is trivial, therefore let $v\in H^1(\Omega)$. By the regularity of Ω, we can fix $\tilde v\in H^1(\Omega)$ an extension of v, then let

$$v_\varepsilon(x)=\begin{cases}v(x) & \text{if } x\in\Omega,\\ \tilde v(x)\left(1-\dfrac{\beta d(x)}{\varepsilon(1+\beta h(x))}\right) & \text{if } x\in\Sigma_\varepsilon,\\ v(x) & \text{if } x\notin\Omega_\varepsilon,\end{cases}$$

where we recall that, if $x=\sigma+t\nu_0(\sigma)$, then $h(x)=h(\sigma)$. Trivially v_ε converges to v in $L^2(\mathbb{R}^n)$ and $v_\varepsilon\in H^1(\Omega_\varepsilon)$. Since $v_\varepsilon=v$ in Ω, we only need to check that

$$\limsup_{\varepsilon\to0^+}\varepsilon\int_{\Sigma_\varepsilon}|\nabla v_\varepsilon|^2\,dx+\beta\int_{\Omega_\varepsilon}v_\varepsilon^2\,d\mathcal{H}^{n-1}\leq\beta\int_{\partial\Omega}\frac{v^2}{1+\beta h}\,d\mathcal{H}^{n-1}. \tag{4.34}$$

For every $x\in\Sigma_\varepsilon$, let

$$\varphi_\varepsilon(x)=1-\frac{\beta d(x)}{\varepsilon(1+\beta h(x))},$$

then, for every $\lambda\in(0,1)$, by convexity

$$\int_{\Sigma_\varepsilon}|\nabla v_\varepsilon|^2\,dx\leq\frac{1}{\lambda}\int_{\Sigma_\varepsilon}\tilde v^2|\nabla\varphi_\varepsilon|^2\,dx+\frac{1}{1-\lambda}\int_{\Sigma_\varepsilon}\varphi_\varepsilon^2|\nabla\tilde v|^2\,dx.$$

Recalling that $0 \le d \le \varepsilon h$, $\nabla d = \nu_0$ and $\nabla h \cdot \nu_0 = 0$, we have

$$|\nabla \varphi_\varepsilon|^2 = \frac{\beta^2}{\varepsilon^2(1+\beta h)^2} + \frac{\beta^4 d^2 |\nabla h|^2}{\varepsilon^2(1+\beta h)^4} \le \frac{\beta^2}{\varepsilon^2(1+\beta h)^2} + \frac{\beta^4 h^2 |\nabla h|^2}{(1+\beta h)^4},$$

where the second term is bounded since h is Lipschitz. Hence, substituting $\varepsilon h \tau = t$ in (3.43), we get

$$\begin{aligned}\varepsilon \int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\,dx &\le \frac{\beta^2}{\varepsilon\lambda} \int_{\Sigma_\varepsilon} \frac{\tilde v^2}{(1+\beta h)^2}\,dx + \frac{\varepsilon}{1-\lambda} \int_{\Sigma_\varepsilon} |\nabla \tilde v|^2\,dx + \varepsilon C \\ &\le \frac{\beta^2}{\lambda} \int_{\partial\Omega} \int_0^1 \frac{\tilde v(\sigma + \varepsilon h \tau \nu_0)^2 h}{(1+\beta h)^2} (1+\varepsilon Q_0)\,d\tau\,d\mathcal{H}^{n-1} + o(\varepsilon).\end{aligned}$$

Passing to the limit, we have that for every $\lambda \in (0,1)$

$$\limsup_{\varepsilon\to 0^+} \varepsilon \int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\,dx \le \frac{\beta^2}{\lambda} \int_{\partial\Omega} \frac{v^2 h}{(1+\beta h)^2}\,d\mathcal{H}^{n-1},$$

hence,

$$\limsup_{\varepsilon\to 0^+} \varepsilon \int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2\,dx \le \beta^2 \int_{\partial\Omega} \frac{v^2 h}{(1+\beta h)^2}\,d\mathcal{H}^{n-1}, \tag{4.35}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$v_\varepsilon(\sigma + \varepsilon h(\sigma)\nu_0(\sigma)) = \frac{\tilde v(\sigma + \varepsilon h(\sigma)\nu_0(\sigma))}{1+\beta h(\sigma)},$$

from which we get, using (3.44)

$$\beta \int_{\partial\Omega_\varepsilon} v_\varepsilon^2\,d\mathcal{H}^{n-1} \le \int_{\partial\Omega} \frac{\beta \tilde v(\sigma + \varepsilon h(\sigma)\nu_0)^2}{(1+\beta h)^2}(1+\varepsilon Q_0)\,d\mathcal{H}^{n-1}.$$

Hence we have

$$\limsup_{\varepsilon\to 0^+} \beta \int_{\partial\Omega_\varepsilon} v_\varepsilon^2\,d\mathcal{H}^{n-1} \le \int_{\partial\Omega} \frac{\beta v^2}{(1+\beta h)^2}\,d\mathcal{H}^{n-1}. \tag{4.36}$$

so that, joining (4.35) and (4.36), we have (4.34) and the Γ-limsup inequality is proved. □

As we extended the energy on the whole $L^2(\mathbb{R}^n)$, minimisers of the problems $\mathcal{E}_{p,\varepsilon}(\cdot, h)$ are not unique; however, in Ω_ε they all coincide with the steady-state temperature $u_\varepsilon \in H^1(\Omega_\varepsilon)$, solution to the boundary value problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega, \\ \Delta u_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ \partial_{\nu_0} u_\varepsilon^- = \varepsilon \partial_{\nu_0} u_\varepsilon^+ & \text{on } \partial\Omega, \\ \partial_\nu u_\varepsilon + \beta u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

However, the extension to $L^2(\mathbb{R}^n)$ is mainly artificial. Indeed, for all intents and purposes, we are only interested in the values the minimisers take in Ω_ε. Hence, in the following, we will assume u_ε extended to zero outside Ω_ε, and refer to it as *the* minimiser of $\mathcal{E}_{p,\varepsilon}(\cdot, h)$. Similarly, we will call *the* minimiser of $\mathcal{E}_{p,0}(\cdot, h)$ the one of

$$\min_{v \in H^1(\Omega)} \mathcal{E}_{p,0}(v, h),$$

that is, the weak solution $u_0 \in H^1(\Omega)$ to the boundary value problem

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ \partial_{\nu_0} u_0 + \dfrac{\beta}{1+\beta h} u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.37}$$

We will also assume u_0 to be zero outside of Ω.

We now prove a Poincaré-type inequality, which will be useful in the following.

Lemma 4.27. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive Lipschitz function $h\colon \partial\Omega \to \mathbb{R}$. Then there exist positive constants $d_0 = d_0(\Omega)$, and $C(\Omega, \|h\|_{C^{0,1}}, \beta,)$ such that if*

$$\varepsilon \|h\|_\infty \leq d_0,$$

then for every $v \in H^1(\Omega_\varepsilon)$

$$\int_{\Omega_\varepsilon} v^2\, dx \leq C\left[\int_\Omega |\nabla v|^2\, dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} v^2\, d\mathcal{H}^{n-1}\right], \tag{4.38}$$

and

$$\int_{\Sigma_\varepsilon} v^2\, dx \leq \varepsilon C\left[\varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} v^2\, d\mathcal{H}^{n-1}\right]. \tag{4.39}$$

Proof. Up to using a density argument, it is enough to prove the assertion for every smooth function in $H^1(\Omega_\varepsilon)$. Let $v \in C^1(\overline{\Omega_\varepsilon})$. For every $x \in \Sigma_\varepsilon$ we recall that we can represent

$$x = \sigma(x) + d(x)\nu_0(x).$$

Extend h as $h(x) = h(\sigma(x))$, and define

$$\xi(x) := \sigma(x) + \varepsilon h(x)\, \nu_0(x).$$

This construction allows us to write for every $x \in \Sigma_\varepsilon$

$$v(x) = v(\xi(x)) - \int_{d(x)}^{\varepsilon h(x)} \frac{\partial}{\partial \nu_0} v(\sigma(x) + t\nu_0(x))\, dt.$$

Hence, by Young and Hölder inequalities,

$$v^2(x) \leq 2v^2(\xi(x)) + 2\varepsilon \|h\|_\infty \int_0^{\varepsilon h(x)} |\nabla v|^2(\sigma(x) + t\nu_0(x))\, dt. \tag{4.40}$$

Integrating over Σ_ε, using (3.45) and (3.46), and recalling that $\xi(x) = \xi(\sigma(x))$, we can find a positive constant $C = C(\Omega, \|h\|_{C^{0,1}})$ such that

$$\begin{aligned} \int_{\Sigma_\varepsilon} v^2\, dx \leq & C \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} v^2(\xi(\sigma))\, ds\, d\mathcal{H}^{n-1}(\sigma) \\ & + \varepsilon C \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} \int_0^{\varepsilon h(\sigma)} |\nabla v|^2(\sigma + t\nu_0)\, dt\, ds\, d\mathcal{H}^{n-1}(\sigma) \\ \leq & \varepsilon C\left(\int_{\partial\Omega_\varepsilon} v^2\, d\mathcal{H}^{n-1} + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\, dx\right). \end{aligned} \tag{4.41}$$

From which we have (4.39). Similarly, we can integrate (4.40) over $\partial\Omega$ and have

$$\int_{\partial\Omega} v^2\, d\mathcal{H}^{n-1} \leq C\left(\int_{\partial\Omega_\varepsilon} v^2\, d\mathcal{H}^{n-1} + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\, dx\right). \tag{4.42}$$

From the Poincaré inequality with trace term in Ω, we have

$$\int_\Omega v^2\,dx \le C_p(\Omega)\Big(\int_\Omega |\nabla v|^2\,dx + \int_{\partial\Omega} v^2\,d\mathcal{H}^{n-1}\Big), \tag{4.43}$$

so that joining (4.41) and (4.43), and using (4.42) we have (4.38). □

Remark 4.28. Let $u_\varepsilon \in H^1(\Omega)$ be the minimisers of $\mathcal{E}_{p,\varepsilon}(\cdot,h)$. Then, by minimality

$$\mathcal{E}_{p,\varepsilon}(u_\varepsilon,h) \le \mathcal{E}_{p,\varepsilon}(0,h) = 0.$$

Hence, we have

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} \le 2\int_\Omega f u_\varepsilon.$$

By standard arguments, using the Poincaré-type inequality (4.38) and Young's inequality, we have that there exists $C = C(\Omega, \|h\|_{C^{0,1}}, \beta, \|f\|_2) > 0$, such that

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} \le C.$$

Then u_ε is bounded in $H^1(\Omega)$ and, up to a subsequence, converges to a function $u \in H^1(\Omega)$ weakly in $H^1(\Omega)$. On the other hand, by (4.39)

$$\int_{\Sigma_\varepsilon} u_\varepsilon^2 \to 0.$$

Extending u to zero outside of Ω we have u_ε converges to u in $L^2(\mathbb{R}^n)$, hence, by the properties of Γ-convergence (see Proposition 2.27) u is a minimiser of the limit energy of $\mathcal{E}_{p,0}(\cdot,h)$ so that u is u_0 and

$$\lim_{\varepsilon\to0^+} \mathcal{E}_{p,\varepsilon}(u_\varepsilon,h) = \mathcal{E}_{p,0}(u_0,h).$$

Finally, we observe that from every subsequence u_{ε_k} we can apply the same argument to prove the convergence of an appropriate subsequence to the same function u_0, hence in particular u_ε converges to u_0 strongly in $L^2(\mathbb{R}^n)$ without needing to pass to a subsequence.

4.2.2 Optimal shape of a thin layer

In this subsection, we study the optimisation problem

$$\min_{(v,h)\in H^1(\Omega)\times\mathcal{H}_m} \mathcal{E}_{p,0}(v,h). \tag{4.44}$$

In particular, in [36], they proved the following theorem.

Theorem 4.29. *Let $f \in L^2(\Omega)$ with $f \ge 0$. For any $m > 0$, there exists $(u,h_0) \in H^1(\Omega)\times L^2(\Omega)$ solution to (4.44). Moreover the couple (u,h_0) is a solution to (4.37) and*

$$h_0(\sigma) = \begin{cases} \dfrac{|u(\sigma)|}{c_u\beta} - \dfrac{1}{\beta} & \text{if } |u(\sigma)| \ge c_u, \\[2mm] 0 & \text{otherwise,} \end{cases} \tag{4.45}$$

where c_u is the unique constant which satisfies

$$c_u = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega\cap\{\,|u|\ge c_u\,\}) + m\beta}\int_{\partial\Omega\cap\{\,|u|\ge c_u\,\}} |u|\,d\mathcal{H}^{n-1}.$$

Finally, if Ω is connected, the couple (u,h_0) is the unique solution to (4.44).

sketch of the proof. The proof of the previous theorem can be divided into 3 main steps that we will briefly discuss in the following.

Step 1. Prove that for every $m > 0$ and every $v \in L^2(\partial\Omega)$ there exists a unique solution $c_v \geq 0$ to the equation

$$c_v = \frac{1}{\mathcal{H}^{n-1}(\{\,|v| \geq c_u\,\}) + m\beta} \int_{\{\,|v|\geq c_u\,\}} |v|\, d\mathcal{H}^{n-1}. \tag{4.46}$$

Then, the function

$$h_v(\sigma) = \begin{cases} \dfrac{|v(\sigma)|}{c_v\beta} - \dfrac{1}{\beta} & \text{if } |u(\sigma)| \geq c_v, \\ 0 & \text{otherwise,} \end{cases}$$

is in $\mathcal{H}_m$ and is the minimiser of

$$\min_{h\in\mathcal{H}_m} \int_{\partial\Omega} \frac{\beta}{1+\beta h} v^2\, d\mathcal{H}^{n-1}.$$

This can be seen using the monotonicity and convexity of the function

$$x \in [0, +\infty) \mapsto \frac{1}{1+x} \in \mathbb{R},$$

and the fact that h_v is stationary for the functional

$$h \in L^1(\partial\Omega) \mapsto \int \frac{\beta}{1+\beta h} v^2\, d\mathcal{H}^{n-1},$$

with the constraint

$$\int_{\partial\Omega} h\, d\mathcal{H}^{n-1} = m.$$

Step 2. Prove that, for every $m > 0$ there exists a constant $C = C(\Omega, m, \beta) > 0$ such that for every $v \in H^1(\Omega)$ and $h \in \mathcal{H}_m$ the following uniform Poincaré-type inequality holds

$$\int_\Omega v^2\, dx \leq C\left[\int_\Omega |\nabla v|^2\, dx + \int_{\partial\Omega} \frac{\beta}{1+\beta h} v^2\, d\mathcal{H}^{n-1}\right]. \tag{4.47}$$

In the paper, this is proved by combining step 1 and the classical trace inequality (see for instance [27, Proposition 2.2] or [17, Proposition 1.5.3])

$$\int_\Omega v^2\, dx \leq C\left[\int_\Omega |\nabla v|^2\, dx + \left(\int_{\partial\Omega} |v|\, d\mathcal{H}^{n-1}\right)^2\right].$$

Indeed, by step 1,

$$\int_\Omega \frac{\beta}{1+\beta h} v^2\, d\mathcal{H}^{n-1} \geq \int_\Omega \frac{\beta}{1+\beta h_v} v^2\, d\mathcal{H}^{n-1} \geq \frac{\beta}{\mathcal{H}^{n-1}(\partial\Omega) + m\beta} \left(\int_{\partial\Omega} |v|^2\, d\mathcal{H}^{n-1}\right)^2,$$

where the last inequality can be obtained by straightforward computation using the definition of h_v.

Then, from (4.47) and standard arguments, one can prove the coerciveness of the energy $\mathcal{E}_{p,0}$, in the sense that there exist two constants $C_1, C_2 > 0$ such that

$$\mathcal{E}_{p,0}(v, h) \geq C_1 \|v\|_{H^1(\Omega)} - C_2,$$

for every $v \in H^1(\Omega)$ and $h \in \mathcal{H}_m$.

Step 3. Given a minimising sequence (u_k, h_k), without loss of generality we can assume that the functions $u_k \in H^1(\Omega)$ are the weak solutions to

$$\begin{cases} -\Delta u_k = f & \text{in } \Omega, \\ \partial_{\nu_0} u_k + \dfrac{\beta}{1+\beta h_k} u_k = 0 & \text{on } \partial\Omega, \end{cases}$$

as they are the minimisers of

$$\min_{v \in H^1(\Omega)} \mathcal{E}_{p,0}(v, h_k).$$

Let $c_k = c_{u_k}$ be defined by the implicit equation (4.46). By the maximum principle, it can be shown that the sequence c_k cannot vanish as k goes to infinity. By the coerciveness in step 2 we have that, up to a subsequence, u_k converges weakly in $H^1(\Omega)$, strongly in $L^2(\partial\Omega)$, to a function $u \in H^1(\Omega)$, and by step 1 we can substitute h_k with

$$h_{u_k} = \begin{cases} \dfrac{u_k}{c_k \beta} - \dfrac{1}{\beta} & \text{if } u_k \geq c_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then h_{u_k} converges in $L^2(\partial\Omega)$ to a function h_0 and (u, h_0) is a minimiser of (4.44) which satisfy (4.37) and (4.45).

Finally, if Ω is connected, the uniqueness is given by the strict convexity of $\mathcal{E}_{p,0}$ in (v, h). $\square$

We now want to look at the proof of Theorem 4.29 from a different perspective. Namely, for every $h \in \mathcal{H}_m$, denote by u_h the solution to the Poisson equation (4.37). We will consider the equivalent problem

$$\min_{h \in \mathcal{H}_m} \mathcal{E}_{p,0}(u_h, h) = \min_{h \in \mathcal{H}_m} -\int_\Omega f u_h \, dx.$$

To prove the existence of a solution, we then need compactness and (semi)continuity results. As we will see in the following, this approach can also be used to study non-variational energies.

We start by defining a new topology in the set $\mathcal{H}_m$. For every $h \in \mathcal{H}_m$, denote by

$$b_h = \frac{\beta}{1+\beta h},$$

the boundary parameter in equation (4.37). By the convexity of the function

$$x \in (0, +\infty) \mapsto \frac{1}{x},$$

we have that the set

$$\mathcal{B}_m(\partial\Omega) = \{\, b_h \colon\ h \in \mathcal{H}_m \,\} = \left\{ b \in L^\infty(\partial\Omega) \;\middle|\; \begin{array}{l} 0 < b \leq \beta \quad \mathcal{H}^{n-1} - \text{a.e. in } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} \frac{1}{b} \, d\mathcal{H}^{n-1} \leq \frac{P(\Omega)}{\beta} + m \end{array} \right\},$$

where we recall that $P(\Omega)$ is the perimeter of Ω, is compact in the the weak-* topology of $L^\infty(\partial\Omega)$.

Definition 4.30. We will say that a sequence $\{h_k\} \subset \mathcal{H}_m$ converges in $\mathcal{H}_m$ to $h \in \mathcal{H}_m$ if and only if the sequence b_{h_k} converges to b_h in the $L^\infty(\partial\Omega)$ weak-* topology.

Remark 4.31. Notice that $\mathcal{H}_m$ with the previously defined topology is compact. We remark that in the original problem in [36] the set $\mathcal{H}_m$ is defined with an *equality* constraint. However, such a constraint is not preserved with the convergence we defined, hence we had to weaken it to an *inequality* constraint.

We have the following proposition

Proposition 4.32. *The function*

$$h \in \mathcal{H}_m \mapsto u_h \in H^1(\Omega)$$

is continuous with respect to the notion of convergence given in Definition 4.30.

Proof. The main tool in proving the continuity is the uniform Poincaré-like inequality (4.47)

$$\int_\Omega v^2\,dx \le C\left[\int_\Omega |\nabla v|^2\,dx + \int_{\partial\Omega} \frac{\beta}{1+\beta h} v^2\,d\mathcal{H}^{n-1}\right].$$

We remark that the previous inequality can also be obtained by a standard contradiction argument and the compactness in the weak-* topology of L^∞ of the set of boundary parameters $\mathcal{B}_m$.

Let $\{ h_k \} \subset \mathcal{H}_m$ such that h_k converges to h in $\mathcal{H}_m$. By standard argument, using the Poincaré inequality (4.47), we can find a constant $C = C(\Omega, m, f)$, independent on k, such that

$$\|u_{h_k}\|_{H^1(\Omega)} \le C$$

for every k. Then, up to a subsequence, u_{h_k} converges weakly in $H^1(\Omega)$ to a function v. We want to prove that $v = u_h$. By (4.37), for every $\varphi \in H^1(\Omega)$ we have that

$$\begin{aligned}\int_\Omega f\varphi\,dx &= \int_\Omega \nabla u_{h_k}\nabla\varphi\,dx + \int_{\partial\Omega} b_{h_k} u_{h_k}\varphi\,d\mathcal{H}^{n-1}\\ &= \int_\Omega \nabla u_{h_k}\nabla\varphi\,dx + \int_{\partial\Omega} b_h u_{h_k}\varphi\,d\mathcal{H}^{n-1} + \int_{\partial\Omega} (b_{h_k} - b_h) u_{h_k}\varphi\,d\mathcal{H}^{n-1}.\end{aligned} \tag{4.48}$$

By the weak convergence of u_{h_k} to v in $H^1(\Omega)$, we have that $u_{h_k}\varphi$ converges strongly in $L^1(\partial\Omega)$ to $v\varphi$, while by definition of convergence in $\mathcal{H}_m$, b_{h_k} converges weakly-* in $L^\infty(\partial\Omega)$ to b_h, hence

$$\lim_{k\to\infty} \int_{\partial\Omega} (b_{h_k} - b_h) u_{h_k}\varphi\,d\mathcal{H}^{n-1} = 0.$$

Then, passing to the limit in (4.48), we have, for every $\varphi \in H^1(\Omega)$,

$$\int_\Omega f\varphi\,dx = \int_\Omega \nabla v\nabla\varphi\,dx + \int_{\partial\Omega} b_h v\varphi\,d\mathcal{H}^{n-1},$$

so that $v = u_h$. Moreover, we have that

$$\int_\Omega |\nabla u_{h_k}|^2\,dx = \int_\Omega f u_{h_k}\,dx - \int_{\partial\Omega} b_{h_k} u_{h_k}^2\,d\mathcal{H}^{n-1}$$

and the right-hand side of the equation converges to

$$\int_\Omega f u_h\,dx - \int_{\partial\Omega} b_h u_h^2\,d\mathcal{H}^{n-1} = \int_\Omega |\nabla u_h|^2\,dx.$$

Then

$$\lim_{k\to\infty} \int_\Omega |\nabla u_{h_k}|^2\,dx = \int_\Omega |\nabla u_h|^2\,dx$$

so that the convergence in $H^1(\Omega)$ is actually strong. Finally, since the limit does not depend on the subsequence, we have that the sequence u_{h_k} converges in $H^1(\Omega)$ to u_h without having to pass to a subsequence. □

Then the existence of a solution (u_{h_0}, h_0) is a direct consequence of the compactness of the set $\mathcal{H}_m$ with respect to the convergence in Definition 4.30 and the continuity result of Proposition 4.32.

As previously observed, the existence result in $\mathcal{H}_m$ does not, in general, guarantee that a solution h_0 saturates the integral constraint. However, if $f \geq 0$ we can say something more. Indeed, we have the following monotonicity property.

Proposition 4.33. *Let $f \geq 0$, then the function*

$$h \in \mathcal{H}_m \mapsto u_h \in H^1(\Omega)$$

is monotonic increasing.

Proof. Let $h_1, h_2 \in \mathcal{H}_m$ such that $h_1 \geq h_2$ and let $w = u_{h_1} - u_{h_2}$. By direct computations, we have that

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \partial_{\nu_0} w + \dfrac{\beta}{1+\beta h_1} w \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Then, by maximum principle, $w \geq 0$ and $u_{h_1} \geq u_{h_2}$. □

Then, if $f \geq 0$, from the previous proposition, we have that the function

$$h \mapsto -\int_\Omega f u_h \, dx$$

is decreasing so that, necessarily, h_0 satisfies

$$\int_{\partial\Omega} h_0 \, d\mathcal{H}^{n-1} = m.$$

Indeed, otherwise, we could contract a better competitor rescaling h_0 as αh_0, for some $\alpha > 1$.

The final step is to characterise the function h_0 as in (4.45). To do that, one may look at the first optimality condition of the problem

$$\min_{h \in \mathcal{H}_m} -\int_\Omega f u_h \, dx.$$

We will outline the idea by working in the set $\mathcal{B}_m$ instead of $\mathcal{H}_m$ as the differentiability of the Robin problem is easier. For every $b \in \mathcal{B}_m$ we will denote by v_b the solution to

$$\begin{cases} -\Delta v_b = f & \text{in } \Omega, \\ \partial_{\nu_0} v_b + b v_b = 0 & \text{on } \partial\Omega, \end{cases}$$

by

$$\mathcal{F}(b) = -\int_\Omega f v_b \, dx$$

and consider the problem

$$\min_{b \in \mathcal{B}_m} \mathcal{F}(b). \tag{4.49}$$

Given a perturbation ψ, denote by $\dot{v}_b = \dot{v}_b[\psi]$ the Gateaux derivative of v_b at b in the direction ψ. We have that

$$\begin{cases} \Delta \dot{v}_b = 0 & \text{in } \Omega \\ \partial_{\nu_0} \dot{v}_b + b\dot{v}_b = -\psi v_b & \text{on } \partial\Omega. \end{cases}$$

Then the first variation of the energy $\mathcal{F}$ is given by

$$\dot{\mathcal{F}}(b)[\psi] = -\int_\Omega f\dot{v}_b[\psi]\,dx = \int_{\partial\Omega} v_b^2\psi\,d\mathcal{H}^{n-1}.$$

If b_0 is a solution to problem (4.49), then by the Kuhn-Tucker theorem (see for instance [53, Theorem 3.2]) there exists Lagrange multipliers $\lambda \geq 0$ and non-negative Radon measure on $\partial\Omega$ μ such that

$$\operatorname{supp}\mu \subseteq \{\, b_0 = \beta \,\},$$

and

$$\dot{\mathcal{F}}(b_0)[\psi] = \lambda\int_{\partial\Omega}\frac{\psi}{b_0^2}\,d\mathcal{H}^{n-1} - \int \psi\,d\mu,$$

for every $\psi \in L^\infty(\partial\Omega)$. Then,

$$v_{b_0}^2 b_0^2 = \lambda \quad \text{in } \{\, b_0 \neq \beta \,\}.$$

Let $c^2\beta^2 = \lambda$, then

$$b_0 = \begin{cases} \dfrac{\beta c}{|v_{b_0}|} & \text{if } |v_{b_0}| > c, \\ \beta & \text{otherwise.} \end{cases}$$

Finally, we can characterise the constant c in terms of v_{b_0}, using the integral constraint of

$$\int_{\partial\Omega}\frac{1}{b_0}\,d\mathcal{H}^{n-1} = \frac{P(\Omega)}{\beta} + m.$$

The original proof of Theorem 4.29 is arguably much more direct and complete than the second approach presented. However, the second framework can be used to prove the existence of a solution to optimisation problems such as maximising the average temperature, that is

$$\max_{h\in\mathcal{H}_m}\int_\Omega u_{h,f}\,dx$$

or minimising the distance from a desired temperature u^*, that is

$$\max_{h\in\mathcal{H}_m}\int_\Omega |u_{h,f} - u^*|^p\,dx,$$

for suitable p (for instance $p \in [1, 2^*]$). More generally, it allows us to study energies of the type

$$\mathcal{F}(h) = \int_\Omega j(x, v_{h,f}, \nabla v_{h,f})\,dx + \int_{\partial\Omega} g(\sigma, v_{h,f}, h)\,d\mathcal{H}^{n-1},$$

for suitable choices of functions j and g.

4.2.3 The case of Dirichlet boundary condition

As mentioned at the beginning of the section, this type of Poisson problem was first introduced for the Dirichlet boundary condition in the context of elastic reinforcement. In [16] and [6], the authors study the limit behaviour of elliptic equations with Dirichlet boundary conditions on a family of sets of the type Ω_ε. The prototype equation is the Poisson problem

$$\begin{cases} -\Delta u_\varepsilon^D = f & \text{in } \Omega, \\ \Delta u_\varepsilon^D = 0 & \text{in } \Sigma_\varepsilon, \\ u_\varepsilon^{D-} = u_\varepsilon^{D+} & \text{on } \partial\Omega, \\ \partial_{\nu_0} u_\varepsilon^{D-} = \varepsilon\partial_{\nu_0} u_\varepsilon^{D+} & \text{on } \partial\Omega, \\ u_\varepsilon^D = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{4.50}$$

In [6], the result is obtained by proving the Γ-convergence of variational energies, while in [16], by more direct computations on the behaviour of the solutions on the set $\partial\Omega$.

In the particular case of problem (4.50), the associated variational energy is

$$\mathcal{E}^D_{p,\varepsilon}(v,h) = \begin{cases} \displaystyle\int_\Omega |\nabla v|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla v|^2\,dx - 2\int_\Omega fv\,dx & \text{if } v \in H^1_0(\Omega_\varepsilon), \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus H^1_0(\Omega_\varepsilon). \end{cases}$$

As ε goes to zero, $\mathcal{E}^D_{p,\varepsilon}(\cdot,h)$ Γ-converges, in the strong $L^2(\mathbb{R}^n)$-topology, to the functional

$$\mathcal{E}^D_{p,0}(v,h) = \begin{cases} \displaystyle\int_\Omega |\nabla v|^2\,dx + \int_{\partial\Omega} \frac{v^2}{h}\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx & \text{if } v \in H^1(\Omega), \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus H^1(\Omega). \end{cases}$$

In particular, the solutions, u^D_ε, to the Poisson problem (4.50) are minimisers of $\mathcal{E}^D_{p,\varepsilon}(\cdot,h)$ and converge, as ε goes to zero, to the function u^D_0, minimiser of $\mathcal{E}^D_{p,0}(\cdot,h)$ and solution to the problem

$$\begin{cases} -\Delta u^D_0 = f & \text{in } \Omega, \\ \partial_{\nu_0} u^D_0 + \dfrac{1}{h}u^D_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.51}$$

We notice that the limit problem (4.51) is exactly the limit, for β going to infinity, of the Poisson problem (4.37) we had in the Robin case. This should not be too surprising since, as already discussed in Section 1.1, the Dirichlet boundary condition can indeed be seen as the limit case of the Robin one. However, it is surprising how, in the limit for ε going to zero, the Dirichlet boundary condition on $\partial\Omega_\varepsilon$ becomes a Robin boundary condition of $\partial\Omega$ instead.

In [27] (see also [18]) the author studies the optimization problem

$$\min_{(v,h)\in H^1(\Omega)\times\mathcal{H}_m} \mathcal{E}^D_{p,0}(v,h), \tag{4.52}$$

which relates to the thermal insulation problem of finding the optimal configuration of insulating material when the main mode of heat transfer with the environment is conduction. As for problem (4.44), problem (5.27) is equivalent to

$$\min_{h\in\mathcal{H}_m} \mathcal{E}^D_{p,0}(u^D_0,h) = \min_{h\in\mathcal{H}_m} -\int_\Omega fu^D_0.$$

On the other hand, for every $v \in L^2(\partial\Omega)\setminus\{0\}$, it is easy to see, by Hölder inequality, that the function

$$h \in \mathcal{H}_m \mapsto \int_{\partial\Omega} \frac{v^2}{h}\,d\mathcal{H}^{n-1}$$

attains its minimum at

$$h^D_v = m\frac{|v|}{\displaystyle\int_{\partial\Omega} |v|\,d\mathcal{H}^{n-1}}.$$

Hence, problem (4.52) is equivalent to the auxiliary problem

$$\min_{v\in H^1(\Omega)} \mathcal{E}^D_{p,0}(v,h_v) = \min_{v\in H^1(\Omega)} \int_\Omega |\nabla v|^2\,dx + \frac{1}{m}\left(\int_{\partial\Omega} |v|\,d\mathcal{H}^{n-1}\right)^2 - 2\int_\Omega fv\,dx. \tag{4.53}$$

By the trace inequality (see for instance [27, Proposition 2.2] or [17, Proposition 1.5.3])

$$\int_\Omega v^2\,dx \le C\left[\int_\Omega |\nabla v|^2\,dx + \left(\int_{\partial\Omega} |v|\,d\mathcal{H}^{n-1}\right)^2\right],$$

and the direct methods of the calculus of variation, the auxiliary problem (4.53) admits a solution, and we have the following theorem.

Theorem 4.34. *Let $f \in L^2(\Omega)$. For any $m \ge 0$, there exists $(u^D, h^D) \in H^1(\Omega) \times \mathcal{H}_m$ solution to (4.52). Moreover, u^D is a solution to the auxiliary problem (4.53) and*

$$h^D = m\frac{|u^D|}{\displaystyle\int_{\partial\Omega} |u^D|\,d\mathcal{H}^{n-1}}.$$

Finally, if Ω is connected, the couple (u^D, h^D) is the unique solution to (4.52).

For every open, bounded, Lipschitz set $\Omega \subset \mathbb{R}^n$, and every $h \in \mathcal{H}_m(\partial\Omega)$ let $w_{\Omega,h}$ be the solution to the limit problem (4.51) when $f = 1$, and let

$$T(\Omega, h) = \int_\Omega w_{\omega,h}\,dx.$$

In analogy with the Saint-Venant inequality, in [18], the authors posed the question of whether, among sets of given measure, the quantity

$$\inf_{h\in\mathcal{H}_m(\partial\Omega)} T(\Omega, h)$$

is maximised by the ball. That is: is it true that among uniformly heated conductors of given volume, the ball-shaped one has the highest mean temperature?

The answer to the question is affirmative. Indeed, for every open, bounded, Lipschitz set $\Omega \subset \mathbb{R}^n$, and every $h \in \mathcal{H}_m(\partial\Omega)$ let $w_{\beta,\Omega,h}$ be the solution to the limit Poisson problem with Robin boundary condition

$$\begin{cases} -\Delta w_{\beta,\Omega,h} = 1 & \text{in } \Omega,\\ \partial_{\nu_0} w_{\beta,\Omega,h} + \dfrac{\beta}{1+\beta h} w_{\beta,\Omega,h} = 0 & \text{on } \partial\Omega, \end{cases}$$

and let

$$T_\beta(\Omega, h) = \int_\Omega w_{\beta,\omega,h}\,dx.$$

In [36], using technique similar to the one in [7] (see also [61]), they proved the following

$$\inf_{h\in\mathcal{H}_m} T_\beta(\Omega, h) \le T_\beta(\Omega^*, m/P(\Omega^*)),$$

where Ω^* is the ball of the same measure as Ω. Passing to the limit as β goes to infinity, $w_{\beta,\omega,h}$ converges to $w_{\omega,h}$, so that

$$\inf_{h\in\mathcal{H}_m} T(\Omega, h) \le T(\Omega^*, m/P(\Omega^*)).$$

4.3 Thin insulating layer: first-order development

The content of this section is based on the results of [3].

We prove a first-order asymptotic development by Γ-convergence (see Definition 2.28) for the energy $\mathcal{E}_{p,\varepsilon}(\cdot, h)$. To study such an asymptotic development, we need to study the behaviour of the minimisers u_ε in the set Σ_ε. As suggested by the recovery sequence in the proof of the Γ-limsup inequality, we expect that the functions u_ε can, in some sense, be approximated by functions that are linear in the normal direction to the boundary of Ω. For the capacitary problem, this was proved by Proposition 3.19 "trapping" the functions between super and sub solutions, for the Poisson problem, instead we "stretch" the solution u_ε via a pullback on the reference set Σ_1 and study the limit as ε goes to zero. To be more precise, we construct a diffeomorphism

$$\Psi_\varepsilon : \Sigma_1 \to \Sigma_\varepsilon$$

by rescaling the distance from $\partial\Omega$. The approach is analogous to the dilation technique proposed in [14] for the asymptotic expansion and the derivation of the so-called effective boundary conditions.

We show the following

Theorem 4.35. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $h(x) = h(\sigma(x))$, and let Ψ_ε be the stretching diffemorphism defined in Definition 4.37. Let u_ε be the minimiser of $\mathcal{E}_{p,\varepsilon}(\cdot, h)$, and let u_0 be the the minimiser of $\mathcal{E}_{p,0}(\cdot, h)$. If we let*

$$\tilde{u}_\varepsilon(z) = \begin{cases} u_\varepsilon(z) & \text{if } z \in \Omega, \\ u_\varepsilon(\Psi_\varepsilon(z)) & \text{if } z \in \Sigma_1, \end{cases}$$

then the family $\tilde{u}_\varepsilon$ is equibounded in $H^1(\Omega_1)$ and it converges weakly in $H^1(\Omega_1)$, as ε goes to 0, to the function

$$\tilde{u}_0(z) = \begin{cases} u_0(z) & \text{if } z \in \Omega, \\ u_0(\sigma(z))\left(1 - \dfrac{\beta d(z)}{1+\beta h(z)}\right) & \text{if } z \in \Sigma_1. \end{cases}$$

Notice that the limit function $\tilde{u}_0$ is indeed linear in the normal direction to the boundary of Ω.

Using Theorem 4.35, we then prove the following first-order development by Γ-convergence.

Theorem 4.36. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $h(x) = h(\sigma(x))$. Then the functional*

$$\delta\mathcal{E}_{p,\varepsilon}(\cdot, h) = \frac{\mathcal{E}_{p,\varepsilon}(\cdot, h) - \mathcal{E}_{p,0}(u_0, h)}{\varepsilon}$$

Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, as $\varepsilon \to 0^+$, to

$$\mathcal{E}_p^{(1)}(v, h) = \begin{cases} \beta \displaystyle\int_{\partial\Omega} \frac{Hh(2+\beta h)}{2(1+\beta h)^2}\, u_0^2 \, d\mathcal{H}^{n-1}, & \text{if } v = u_0, \\ +\infty & \text{if } v \neq u_0, \end{cases}$$

where H denotes the mean curvature of $\partial\Omega$.

In subsection 4.3.1, we compute the diffeomorphism ψ_ε and study the limit equation in the reference set Σ_1. In subsection 4.3.2 we prove H^2-energy estimates that will be crucial in the proof of Theorem 4.35. In subsection 4.3.3 we prove Theorem 4.35 and Theorem 4.36

4.3.1 Preliminary computations

In this subsection, we construct the diffeomorphism $\Psi_\varepsilon \colon \Sigma_1 \to \Sigma_\varepsilon$, and, assuming that the family of functions $\tilde{u}_\varepsilon = u_\varepsilon \circ \Psi_\varepsilon$ converges weakly in $H^1(\Sigma_1)$ to a function $\tilde{u}_0$, we compute the limiting boundary value problem for the limit $\tilde{u}_0$.

The stretching diffemorphism

Let Ω be a bounded open set with $C^{1,1}$ boundary. For every $x \in \Gamma_{d_0}$, we recall we have defined

$$\nu_0(x) = \nu_0(\sigma(x))$$

to be the unit outer normal to $\partial\Omega$ in $\sigma(x)$, and let $h \in C^{1,1}(\Gamma_{d_0})$ be a positive function such that $h(x) = h(\sigma(x))$, so that h is constant along normal radii starting from $\partial\Omega$. We define

$$\Sigma_\varepsilon := \{\, x \in \mathbb{R}^n \mid 0 < d(x) < \varepsilon h(x) \,\}, \qquad \Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon.$$

In particular, notice that, if $\varepsilon\|h\|_\infty < d_0$, we can write

$$\Sigma_\varepsilon = \left\{ \sigma + t\nu_0(\sigma) \;\middle|\; \begin{array}{c} \sigma \in \partial\Omega, \\ 0 < t < \varepsilon h(\sigma) \end{array} \right\}, \tag{4.54}$$

and the representation $x = \sigma + t\nu_0$ is unique and $d \in C^{1,1}(\Sigma_\varepsilon)$.

For simplicity's sake, we will assume that $\|h\|_\infty < d_0$ so that we can assume the distance function d to be regular on the set Σ_1.

Definition 4.37. We define the *stretching diffeomorphism* $\Psi_\varepsilon \in C^{0,1}(\Gamma_{d_0}; \Gamma_{\varepsilon d_0})$ as the function defined as

$$\begin{aligned} \Psi_\varepsilon(z) &= \sigma(z) + \varepsilon d(z)\nu_0(z) \\ &= z + (\varepsilon - 1)d(z)\nu_0(z). \end{aligned}$$

With this definition, we have that

$$\begin{aligned} D\Psi_\varepsilon(z) &= D\sigma(z) + \varepsilon[\nu_0(z) \otimes \nu_0(z) + d(z)D\nu_0(z)] \\ &= \mathrm{Id}_n + (\varepsilon - 1)[\nu_0(z) \otimes \nu_0(z) + d(z)D\nu_0(z)], \end{aligned}$$

where Id_n is the identity matrix. Moreover, Ψ_ε is invertible with

$$\Psi_\varepsilon^{-1}(x) = \sigma(x) + \frac{d(x)}{\varepsilon}\nu_0(x).$$

By direct computations, we have the following

Lemma 4.38. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $h(x) = h(\sigma(x))$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive Borel function. Then*

$$\int_{\Sigma_\varepsilon} g(x)\,dx = \varepsilon \int_{\Sigma_1} g(\Psi_\varepsilon(z))J_\varepsilon(z)\,dz, \tag{4.55}$$

with

$$J_\varepsilon(z) = \prod_{i=1}^{n-1} \frac{1 + \varepsilon d(z)k_i(\sigma)}{1 + d(z)k_i(\sigma)},$$

and $\sigma = \sigma(z)$. In addition,

$$\int_{\partial\Omega_\varepsilon} g(\xi)\,d\mathcal{H}^{n-1}(\xi) = \int_{\partial\Omega_1} g(\Psi_\varepsilon(\zeta))J_\varepsilon^\tau(\zeta)\,d\mathcal{H}^{n-1}(\zeta), \tag{4.56}$$

where J^τ_ε is the tangential Jacobian of Ψ_ε, and it converges uniformly, as $\varepsilon \to 0^+$, to

$$\begin{aligned} J^\tau_0(\zeta) &= \frac{1}{\sqrt{1+|\nabla h(\zeta)|^2}} \prod_{i=1}^{n-1} \frac{1}{1+h(\zeta)k_i(\sigma(\zeta))} \\ &= \frac{1}{\sqrt{1+|\nabla h(\zeta)|^2}} J_0(\zeta). \end{aligned}$$

Proof. Let $z \in \Gamma_{d_0}$, and let $\tau_i(z)$ be the eigenvectors of $D^2 d(z) = D\nu_0(z)$, with respective eigenvectors $k_i(z)$, defined in Definition 2.17. We have that $\{ \tau_1(\sigma), \dots, \tau_{n-1}(\sigma), \nu_0(\sigma) \}$ is a basis of eigenvectors for $D\Psi_\varepsilon(z)$, and, in particular,

$$\begin{aligned} D\Psi_\varepsilon(z)\,\tau_i(z) &= \left(1 + \frac{(\varepsilon-1)d(z)k_i(\sigma)}{1+d(z)k_i(\sigma)}\right)\tau_i(z) \\ &= \frac{1+\varepsilon d(z)k_i(\sigma)}{1+d(z)k_i(\sigma)}\tau_i(z), \end{aligned}$$

and

$$D\Psi_\varepsilon(z)\,\nu_0(z) = \varepsilon\nu_0(z).$$

Therefore, (4.55) follows from the area formula.

On the other hand, notice that $D\Psi_\varepsilon$ converges uniformly, as $\varepsilon \to 0^+$, to $D\sigma$, so that $J^{\partial\Omega_1}\Psi_\varepsilon$ converges to $J^{\partial\Omega_1}\sigma$. Therefore, we apply the area formula on surfaces (Theorem 2.21) to get (4.56), and it is left to compute the tangential Jacobian $J^{\partial\Omega_1}\sigma$.

Let $\zeta \in \partial\Omega_1$, and let ν_1 be the unit outer normal to $\partial\Omega_1$. Recalling that $\nabla d = \nu_0$, the definition of Σ_1 ensures that

$$\nu_1(\zeta) = \frac{\nu_0(\zeta) - \nabla h(\zeta)}{\sqrt{1+|\nabla h(\zeta)|^2}}.$$

We aim to construct an orthonormal basis for the tangent space $T_\zeta\partial\Omega_1$ with a rotation of the tangent space $T_{\sigma(\zeta)}\partial\Omega$. In the following, when possible, we will drop the dependence on ζ. Let us define the rotation operator

$$R : \mathbb{R}^n \to \mathbb{R}^n$$

as the unique linear operator having the following properties:

(i) if $w \in (\nabla h)^\perp \cap \nu_0^\perp$, then $Rw = w$;

(ii) $R\nu_0 = \nu_1$;

(iii) $\nabla h/|\nabla h|$ is mapped to a unit vector in the plane generated by ∇h and ν_0, and orthogonal to ν_1, namely

$$R\frac{\nabla h}{|\nabla h|} = \frac{1}{\sqrt{1+|\nabla h|^2}}\left(\frac{\nabla h}{|\nabla h|} + |\nabla h|\nu_0\right).$$

Explicitly, we can write R as

$$\begin{aligned} R = \mathrm{Id}_n - \nu_0 \otimes \nu_0 - \frac{\nabla h}{|\nabla h|} \otimes \frac{\nabla h}{|\nabla h|} + \nu_1 \otimes \nu_0 \\ + \frac{1}{\sqrt{1+|\nabla h|^2}}\left(\frac{\nabla h}{|\nabla h|} + |\nabla h|\nu_0\right) \otimes \frac{\nabla h}{|\nabla h|}. \end{aligned}$$

This operator is a rotation in $\mathbb{R}^n$ that maps $T_{\sigma(\zeta)}\partial\Omega$ onto $T_\zeta\partial\Omega_1$, and, in particular, we define an orthonormal basis $\bar\tau_i$ of $T_\zeta\partial\Omega_1$ as follows: let $\tau_i = \tau_i(\sigma(\zeta))$ be an orthonormal basis of $T_{\sigma(\zeta)}\partial\Omega$, and we define

$$\bar\tau_s := R\tau_s = \left(\mathrm{Id}_n + \left(\frac{1}{\sqrt{1+|\nabla h|^2}} - 1\right)\frac{\nabla h}{|\nabla h|}\otimes\frac{\nabla h}{|\nabla h|}\right)\tau_s + \frac{\nabla h\cdot\tau_s}{\sqrt{1+|\nabla h|^2}}\,\nu_0.$$

In particular, since $D\sigma\,\nu_0 = 0$, then

$$(D\sigma\circ R)\tau_s = D\sigma\left(\mathrm{Id}_n + \left(\frac{1}{\sqrt{1+|\nabla h|^2}} - 1\right)\frac{\nabla h}{|\nabla h|}\otimes\frac{\nabla h}{|\nabla h|}\right)\tau_s.$$

Hence, recalling the evaluation of the eigenvalues of $D\sigma$ given in Remark 2.18, we can easily compute the determinant of the tangential gradient

$$J^{\partial\Omega_1}\sigma(\zeta) = \left(\prod_{i=1}^{n-1}(1+h(\zeta)k_i(\sigma(\zeta)))\sqrt{1+|\nabla h(\zeta)|^2}\right)^{-1},$$

thus concluding the proof. □

The limit equation

For every $0<\varepsilon<1$ let $u_\varepsilon \in H^1(\Sigma_\varepsilon)$ be a solution to

$$\varepsilon\int_{\Sigma_\varepsilon}\nabla u_\varepsilon\nabla\varphi\,dx + \beta\int_{\partial\Omega_\varepsilon}u_\varepsilon\varphi\,d\mathcal{H}^{n-1} = 0, \tag{4.57}$$

for every $\varphi\in H^1(\Sigma_\varepsilon)$ such that $\varphi = 0$ on $\partial\Omega$. Then we have the following.

Proposition 4.39. *Let $\Omega\subset\mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h\in C^{1,1}(\Gamma_{d_0})$ such that $h(x) = h(\sigma(x))$. Let u_ε be a family of weak solution to (4.57), and let $\tilde u_\varepsilon(z) = u_\varepsilon(\Psi_\varepsilon(z))$. If $\tilde u_\varepsilon$ converges weakly in $H^1(\Sigma_1)$ to a function $\tilde u_0$, then $\tilde u_0$ is a solution to*

$$\int_{\Sigma_1}(\nabla\tilde u_0\cdot\nu_0)(\nabla\varphi\cdot\nu_0)J_0(z)\,dz + \beta\int_{\partial\Omega_1}\tilde u_0(\zeta)\varphi(\zeta)\frac{J_0(\zeta)}{\sqrt{1+|\nabla h(\zeta)|^2}}\,d\mathcal{H}^{n-1}(\zeta) = 0. \tag{4.58}$$

for every $\varphi\in H^1(\Sigma_1)$ such that $\varphi = 0$ on $\partial\Omega$.

Proof. By definition, we have $\tilde u_\varepsilon\in H^1(\Sigma_1)$ and

$$\begin{aligned}\nabla u_\varepsilon|_{\Psi_\varepsilon(z)} &= D(\Psi_\varepsilon^{-1})|_{\Psi_\varepsilon(z)}\nabla\tilde u_\varepsilon(z)\\ &= \left(\frac{1}{\varepsilon}\nabla\tilde u_\varepsilon(z)\cdot\nu_0(z)\right)\nu_0(z) + \sum_{i=1}^{n-1}\left(\frac{1+d(z)k_i(\sigma)}{1+\varepsilon d(z)k_i(\sigma)}\nabla\tilde u_\varepsilon(z)\cdot\tau_i(z)\right)\tau_i(z).\end{aligned}\tag{4.59}$$

Let $\varphi\in H^1(\Sigma_1)$ with $\varphi = 0$ on $\partial\Omega$, and let $\varphi_\varepsilon(x) = \varphi(\Psi_\varepsilon^{-1}(x))$, then equation (4.57) yields

$$\varepsilon\int_{\Sigma_\varepsilon}\nabla u_\varepsilon\nabla\varphi_\varepsilon\,dx + \beta\int_{\partial\Omega_\varepsilon}u_\varepsilon\varphi_\varepsilon\,d\mathcal{H}^{n-1} = 0,$$

from which, using Lemma 4.38 and the computation (4.59), we have

$$\begin{aligned}\int_{\Sigma_1}\left[(\nabla\tilde u_\varepsilon\cdot\nu_0)(\nabla\varphi\cdot\nu_0) + \varepsilon^2\sum_{i=1}^{n-1}\left(\frac{1+dk_i}{1+\varepsilon dk_i}\right)^2(\nabla\tilde u_\varepsilon\cdot\tau_i)(\nabla\varphi\cdot\tau_i)\right]\prod_{i=1}^{n-1}\frac{1+\varepsilon dk_i}{1+dk_i}\,dz+\\ +\beta\int_{\partial\Omega_1}\tilde u_\varepsilon\varphi J_\varepsilon^\tau\,d\mathcal{H}^{n-1} = 0,\end{aligned}\tag{4.60}$$

Passing to the limit in (4.60), the assertion follows. □

Remark 4.40 (Uniqueness)**.** For every given Dirichlet boundary condition on $\partial\Omega$, (4.58) admits a unique solution. Indeed, let $v_1, v_2 \in H^1(\Sigma_1)$ be two solutions to (4.58) such that $v_1 = v_2$ on $\partial\Omega$, and let $w = v_1 - v_2$. By linearity, we have that w is a solution to (4.58) with $w = 0$ on $\partial\Omega$, so that we have

$$\int_{\Sigma_1} |\nabla w \cdot \nu_0|^2 J_0 \, dz + \beta \int_{\partial\Omega_1} w^2 \frac{J_0}{\sqrt{1 + |\nabla h|^2}} \, d\mathcal{H}^{n-1} = 0,$$

and since $J_0 > 0$, then $\nabla w \cdot \nu_0 = 0$ a.e. on Σ_ε. Then, for $\mathcal{L}^n$-a.e. $z \in \Sigma_1$,

$$w(z) = w(\sigma(z)) + \int_0^{d(z)} \nabla w(\sigma(z) + t\nu_0) \cdot \nu_0 \, dt = 0,$$

that is $w = 0$ and $v_1 = v_2$.

Remark 4.41 (Limit computation)**.** We point out that it is possible to explicitly compute the solution $\tilde{u}_0$ to (4.58) in terms of its values on $\partial\Omega$ as

$$\tilde{u}_0(z) = \tilde{u}_0(\sigma(z)) \left(1 - \frac{\beta d(z)}{1 + \beta h(z)} \right).$$

We first inspect the regular case of Ω of class C^3 to obtain the strong equation, and then we work on the general case using only the weak equation (4.58).

Indeed, let

$$A(z) = J_0(z) \nu_0(z) \otimes \nu_0(z),$$

and notice that when Ω is smooth, then J_0 is also smooth, as it can be seen as the determinant of the smooth matrix-valued map

$$z \in \Sigma_1 \mapsto D\sigma + \nu_0 \otimes \nu_0.$$

Then, if $\tilde{u}_0$ is a regular solution to (4.58), it is a solution to

$$\begin{cases} \operatorname{div}(A(x)\nabla \tilde{u}_0) = 0 & \text{in } \Sigma_1, \\ \dfrac{\partial \tilde{u}_0}{\partial \nu_0} + \beta \tilde{u}_0 = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{4.61}$$

Moreover, we have that

$$\operatorname{div}(J_0 \nu_0) = \nabla J_0 \cdot \nu_0 + J_0 \operatorname{Tr}(D\nu_0).$$

In particular, we can explicitly compute the derivative of J_0 in direction ν_0 using the local representation

$$J_0(z) = \prod_{i=1}^{n-1} \frac{1}{1 + d(z) k_i(\sigma(z))},$$

and recalling that $k_i \circ \sigma$ is constant along normal radii, so that, for every $\zeta \in \partial\Omega_1$,

$$\nabla J_0(\zeta) \cdot \nu_0(\zeta) = -J_0(\zeta) \operatorname{Tr}(D\nu_0(\zeta)).$$

Hence, $\operatorname{div}(J_0 \nu_0) = 0$, and

$$\operatorname{div}(A\nabla u) = J_0 \nabla(\nabla u \cdot \nu_0) \cdot \nu_0.$$

Therefore, equation (4.61) reduces to

$$\begin{cases} \nabla(\nabla \tilde{u}_0 \cdot \nu_0) \cdot \nu_0 = 0 & \text{in } \Sigma_1, \\ \dfrac{\partial \tilde{u}_0}{\partial \nu_0} + \beta \tilde{u}_0 = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{4.62}$$

The previous computation suggests that solutions to (4.58) have to be linear with respect to the normal direction. Indeed, since h is constant along the normal direction, it is sufficient to check that, for every $w \in H^1(\Sigma_1)$, the function

$$\tilde{u}(z) = w(\sigma(z))\left(1 - \frac{\beta d(z)}{1 + \beta h(z)}\right)$$

is a solution to (4.62) in $H^1(\Sigma_1)$. Finally, we can show that the previous solution to (4.62) is the solution to (4.58) also in the case in which Ω is only $C^{1,1}$. By a change of variables and coarea formula (see Remark 3.16) we can rewrite the integral on Σ_1 as

$$\int_{\Sigma_1} g(z) J_0(z)\, dz = \int_{\partial\Omega} \int_0^{h(\sigma)} g(\sigma + t\nu_0)\, dt\, d\mathcal{H}^{n-1}(\sigma)$$

and the integral on $\partial\Omega_1$ as

$$\int_{\partial\Omega_1} g(\xi) \frac{J_0(\xi)}{\sqrt{1 + |\nabla h|^2}}\, d\mathcal{H}^{n-1}(\xi) = \int_{\partial\Omega} g(\sigma + h(\sigma)\nu_0)\, d\mathcal{H}^{n-1}(\sigma).$$

Then, by direct computation, we have that

$$\nabla\tilde{u} \cdot \nu_0 = -\frac{\beta w(\sigma(z))}{1 + \beta h(z)}$$

so that, for every smooth function $\varphi \in H^1(\Sigma_1)$ with $\varphi = 0$ on $\partial\Omega$,

$$\begin{aligned}
\int_{\Sigma_1} (\nabla\tilde{u}_0 \cdot \nu_0)(\nabla\varphi \cdot \nu_0) J_0(z)\, dz &= -\int_{\partial\Omega} \frac{\beta w(\sigma}{1 + \beta h} \int_0^{h(\sigma)} \frac{d}{dt}(\varphi(\sigma + t\nu_0))\, dt\, d\mathcal{H}^{n-1}(\sigma) \\
&= -\int_{\partial\Omega} \frac{\beta w(\sigma}{1 + \beta h} \int_0^{h(\sigma)} \varphi(\sigma + h(\sigma)\nu_0)\, d\mathcal{H}^{n-1} \\
&= -\beta \int_{\partial\Omega} \tilde{u}_0(\sigma + h(\sigma)\nu_0)\varphi(\sigma + h(\sigma)\nu_0)\, d\mathcal{H}^{n-1} \\
&= -\beta \int_{\partial\Omega_1} \tilde{u}_0(\zeta)\varphi(\zeta) \frac{J_0(\zeta)}{\sqrt{1 + |\nabla h(\zeta)|^2}}\, d\mathcal{H}^{n-1}(\zeta)
\end{aligned}$$

and $\tilde{u}$ is in fact a solution to (4.58).

4.3.2 Energy estimates

Recall that the minimiser u_ε of $\mathcal{E}_{p,\varepsilon}(\cdot, h)$ is the solution to

$$\int_\Omega \nabla u_\varepsilon \nabla\varphi\, dx + \varepsilon \int_{\Sigma_\varepsilon} \nabla u_\varepsilon \nabla\varphi\, dx + \beta \int_{\partial\Omega} u_\varepsilon \varphi\, d\mathcal{H}^{n-1} = \int_\Omega f\varphi\, dx, \tag{4.63}$$

For every $\varphi \in H^1(\Omega_\varepsilon)$. In this section, we prove the following theorem.

Theorem 4.42. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ and $h(x) = h(\sigma(x))$. Then there exists positive constants $\varepsilon_0(\Omega)$, and $C(\Omega, h, \beta, f)$ such that if*

$$\varepsilon \|h\|_{C^{0,1}} \le \varepsilon_0 \tag{4.64}$$

and u_ε is the minimiser of $\mathcal{E}_{p,\varepsilon}(\cdot, h)$, then

$$\int_\Omega |D^2 u_\varepsilon|^2\, dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} |\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2\, d\mathcal{H}^{n-1} \le C. \tag{4.65}$$

Remark 4.43. We want to point out that the assumption $h(x) = h(\sigma(x))$ is not necessary to prove Theorem 4.42, but it makes the computations easier.

As observed in Remark 4.28, from Lemma 4.27 we have the following H^1 estimates

Corollary 4.44. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{0,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$. Then there exists a positive constant $C = C(\Omega, h, \beta, f)$ such that if $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is the minimiser of $\mathcal{E}_{p,\varepsilon}(\cdot, h)$, then*

$$\int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} \le C, \tag{4.66}$$

and

$$\int_{\Omega_\varepsilon} u_\varepsilon^2\,dx \le C. \tag{4.67}$$

Proof. For every $\eta > 0$, we can write

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} &\le \mathcal{E}_{p,\varepsilon}(0) + 2\int_\Omega f u_\varepsilon\,dx \\ &\le \eta \int_\Omega f^2\,dx + \frac{1}{\eta}\int_\Omega u_\varepsilon^2\,dx.\end{aligned}$$

Using (4.38), for a suitable choice of η, we get the result. ☐

To prove Theorem 4.42, we use a local argument similar to the approach in [16]: we focus on small neighbourhoods V_{σ_0} of points $\sigma_0 \in \partial\Omega$, and we construct a diffeomorphism Φ_{σ_0} that flattens both $\partial\Omega$ and $\partial\Omega_\varepsilon$; on the flattened set $\Phi(V_{\sigma_0} \cap \Sigma_\varepsilon)$ we can compute energy estimates of the new functions $v_\varepsilon = u_\varepsilon \circ \Phi^{-1}$.

Flattening the boundaries

Here we aim to construct a flattening diffeomorphism that locally transforms $\partial\Omega$ and $\partial\Omega_\varepsilon$ in subsets of parallel planes. To do so, we have to represent locally $\partial\Omega$ and $\partial\Omega_\varepsilon$.

Lemma 4.45 (Uniform local representation of $\partial\Omega$ and $\partial\Omega_\varepsilon$.)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, fix a positive function $h \in C^{0,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$, and let $\sigma_0 \in \partial\Omega$. There exists $\varepsilon_0 = \varepsilon_0(\Omega, \sigma_0)$ such that, if*

$$\varepsilon \|h\|_{C^{0,1}} < \varepsilon_0,$$

then there exist an open set V containing σ_0 and $\sigma_0 + \varepsilon h(\sigma_0)\nu_0(\sigma_0)$, and there exist functions $g, k_\varepsilon \colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that, up to a rototranslation,

$$\begin{aligned}\Omega \cap V &= \left\{\, (x', x_n) \;\middle|\; x_n \le g(x') \,\right\} \cap V, \\ \Omega_\varepsilon \cap V &= \left\{\, (x', x_n) \;\middle|\; x_n \le g(x') + \varepsilon k_\varepsilon(x') \,\right\} \cap V,\end{aligned}$$

and

$$\begin{aligned}\partial\Omega \cap V &= \left\{\, (x', x_n) \;\middle|\; x_n = g(x') \,\right\} \cap V, \\ \partial\Omega_\varepsilon \cap V &= \left\{\, (x', x_n) \;\middle|\; x_n = g(x') + \varepsilon k_\varepsilon(x') \,\right\} \cap V,\end{aligned}$$

Proof. Without loss of generality we can assume that $\nu_0(\sigma_0) = \mathbf{e}_n$ and $\sigma_0 = 0$. We already know by definition that Ω can be represented locally near 0, that is, there exist a neighbourhood U of σ_0 and a function $g \in C^{1,1}(\mathbb{R}^{n-1})$ with $g(0) = 0$ and $\nabla g(0) = 0$, such that

$$\Omega \cap U = \left\{\, (x', x_n) \in \mathbb{R}^n \;\middle|\; x_n \le g(x') \,\right\} \cap U.$$

For every $r \in (0,1)$ let

$$B'_r = \left\{ x' \in \mathbb{R}^{n-1} \;\middle|\; |x'| \le r \right\},$$

and $V_r = B'_r \times [-2r, 2r]$. For every $x' \in B'_r$ let

$$F_{x'} : t \in (g(x'), 2r] \longmapsto d(x', t) - \varepsilon h(x', t),$$

where we recall that $d(x)$ denotes the distance from Ω. Let us also recall that

$$\Sigma_\varepsilon = \Omega_\varepsilon \setminus \overline{\Omega} = \{ x \in \mathbb{R}^n \mid 0 < d(x) < \varepsilon h(x) \}.$$

The definition of $F_{x'}$ gives us the possibility to characterise the property $(x', t) \in \Omega_\varepsilon$ in the equivalent way $F_{x'}(t) < 0$. As an immediate consequence, $F_{x'}(g(x')) < 0$ for every $x' \in B'_r$.

The idea of the proof can be divided in two main steps: first we show that for a right choice of ε_0 the set $\partial\Omega_\varepsilon$ cannot touch the upper base of the cylinder V_r by proving $F_{x'}(2r) > 0$ for every $x' \in B'_r$; next, we show that for a right choice of ε_0 we have that $\partial\Omega_\varepsilon$ can be represented as a graph over the whole B'_r by showing that $F_{x'}$ is strictly increasing in t for every $x' \in B'_r$.

Let us choose $r = r(\sigma_0, \Omega)$ small enough so that we have $V_r \subseteq U$ and, thanks to the continuity of ν_0 in Γ_{d_0}

$$|\nu_0(x) - \mathbf{e}_n| < \frac{1}{2}, \tag{4.68}$$

for every $x \in V_r \cap \Gamma_{d_0}$. Note in addition that up to choosing a smaller r we can assume the vertical distance between $\partial\Omega$ and the upper base of V_r is greater than r, namely

$$2r - \max_{x' \in B'_r} g(x') > r. \tag{4.69}$$

Indeed, since ∇g is continuous and $\nabla g(0) = 0$, then for small enough r we also have, for every $x' \in B'_r$,

$$|\nabla g(x')| < 1,$$

which joint with the fact that $g(0) = 0$, ensures (4.69).

Let $x' \in B'_r$, fix $t \in (g(x'), 2r]$, and let $x = (x', t)$. We claim that under the assumption (4.68), we have

$$d(x) > \frac{t - g(x')}{2}. \tag{4.70}$$

Indeed, let $\bar{x}$ denote the point at distance $d(\bar{x}) = t - g(x')$ whose projection onto Ω is $(x', g(x'))$, namely

$$\overline{x} = (x', g(x')) + (t - g(x'))\nu_0(x', g(x')).$$

We have

$$|x - \overline{x}| = (t - g(x'))\,|\mathbf{e}_n - \nu_0(x', g(x'))| < \frac{t - g(x')}{2},$$

and, using the fact that the distance d is Lipschitzian with constant 1,

$$\begin{aligned} d(x) &\ge d(\overline{x}) - |d(x) - d(\overline{x})| \\ &\ge d(\overline{x}) - |x - \overline{x}| > \frac{t - g(x')}{2}. \end{aligned}$$

We now join (4.70), (4.69), and $\varepsilon\|h\|_\infty < \varepsilon_0$ to get that for $\varepsilon_0 < r/2$

$$F_{x'}(2r) > \frac{2r - g(x')}{2} - \varepsilon_0 > 0, \tag{4.71}$$

which concludes the first step.

We now prove that $F_{x'}(t)$ is monotone increasing in t. Indeed, by the assumption (4.68), we have that

$$\nu_0(x)\cdot \mathbf{e}_n > \frac{1}{2},$$

and

$$|\nabla h(x)\cdot \mathbf{e}_n| = |\nabla h(x)\cdot(\mathbf{e}_n - \nu_0(x))| \le \frac{\|\nabla h\|_\infty}{2},$$

so that, since $r<1$ and $\varepsilon_0 < r/2$, we have $\varepsilon\|\nabla h\|_\infty < 1/2$, which ensures

$$\frac{d}{dt}F_{x'}(t) = \nu_0(x',t)\cdot \mathbf{e}_n - \varepsilon\nabla h(x',t)\cdot\mathbf{e}_n > \frac{1}{4}. \tag{4.72}$$

Joining (4.71) and (4.72), we get that for every $x'\in B_r'$ there exists a unique $t(x')\in(g(x'),2r)$ such that $(x',t)\in\Omega_\varepsilon\cap V_r$ if and only if $t\le t(x')$, thus concluding the proof. □

Thanks to Lemma 4.45, we can now represent both $\partial\Omega$ and $\partial\Omega_\varepsilon$ as graphs, uniformly in ε, and flatten the boundaries of Ω and Ω_ε. We define the invertible map

$$\Phi_{\sigma_0}\colon (x',x_n)\in V \mapsto \left(x', \frac{x_n - g(x')}{k_\varepsilon(x')}\right)\in \tilde V,$$

where $\tilde V = \Phi_{\sigma_0}(V)$. For simplicity's sake, when possible, we will drop the explicit dependence on the point $\sigma_0\in\partial\Omega$. Notice that the map Φ indeed flattens the boundaries of Ω and Ω_ε, in the sense that

$$\Phi(\partial\Omega\cap V) = \{\, y_n = 0\,\}\cap\tilde V,$$

and

$$\Phi(\partial\Omega_\varepsilon\cap V) = \{\, y_n = \varepsilon\,\}\cap\tilde V.$$

For every $\delta>0$ we define the cube

$$\tilde Q_\delta = \{\, y\in\mathbb{R}^n \mid |y_i|\le\delta, \text{ for every } 1\le i\le n\,\}.$$

Up to choosing a smaller neighborhood V of σ_0, we may assume that

$$\tilde V = \Phi(V) = \tilde Q_{\delta_0}$$

for some $\delta_0 = \delta_0(\sigma_0,\Omega)>0$. For every $0<\delta<\delta_0$ we define

$$Q_\delta = \Phi^{-1}(\tilde Q_\delta).$$

Remark 4.46 (Estimates for k_ε, Φ, and Φ^{-1})**.** We claim that there exists a positive constant $C = C(\Omega,\|h\|_{C^{1,1}})$ such that

$$\min k_\varepsilon \ge \frac{1}{C}, \qquad \|k_\varepsilon\|_{C^{1,1}} + \|\Phi\|_{C^{1,1}} + \|\Phi^{-1}\|_{C^{1,1}} \le C. \tag{4.73}$$

Notice that, since $\Omega\subset\Omega_\varepsilon$ for every $\varepsilon>0$, then k_ε is non-negative. Moreover, for every $x=(x',x_n)\in\partial\Omega_\varepsilon\cap V$

$$\varepsilon k_\varepsilon(x') \ge d(x,\partial\Omega)\ge \varepsilon\min_{\Gamma_{d_0}} h,$$

so that k_ε is strictly bounded from below uniformly in ε. Similarly, k_ε is bounded in $C^{1,1}$ norm uniformly in ε, indeed for every $x\in\partial\Omega_\varepsilon\cap V$, we can write $x=(x',g(x')+\varepsilon k_\varepsilon(x'))$, and we have that

$$d(x,\partial\Omega_\varepsilon) = 0.$$

Differentiating with respect to x' we get

$$\nu_\varepsilon' + \nu_\varepsilon\cdot\mathbf{e}_n(\nabla g + \varepsilon\nabla k_\varepsilon) = 0, \tag{4.74}$$

where $\nu'_\varepsilon \in \mathbb{R}^{n-1}$ is the vector whose components are the first $n-1$ components of ν_ε. Using the fact that

$$\nu_\varepsilon = \frac{\nu_0 - \varepsilon\nabla h}{\sqrt{1+\varepsilon^2|\nabla h|^2}},$$

and that in V

$$\nu_0 = \left(-\frac{\nabla g}{\sqrt{1+|\nabla g|^2}}, \frac{1}{\sqrt{1+|\nabla g|^2}}\right),$$

we can rewrite equation (4.74) as

$$\nabla k_\varepsilon = \frac{(\nabla h)' + (\nabla h\cdot \mathbf{e}_n)\nabla g}{(\nu_0 - \varepsilon\nabla h)\cdot \mathbf{e}_n}. \tag{4.75}$$

Finally, the choice of ε_0 in Lemma 4.45 (see (4.72)) ensures us that

$$(\nu_0 - \varepsilon\nabla h)\cdot \mathbf{e}_n > \frac{1}{4}, \tag{4.76}$$

so that from equation (4.75) and (4.76), (4.73) follows.

Remark 4.47 (The equation in the flattened set)**.** Let u_ε be the solution to (4.63), fix $\sigma_0 \in \partial\Omega$, and let

$$v(y) = u_\varepsilon(\Phi^{-1}(y)),$$

then

$$v \in H^1\big(\{\, y_n < \varepsilon \,\}\cap \tilde{V}\big) \cap H^2_{\mathrm{loc}}\Big(\big(\{\, y_n < \varepsilon \,\}\cap \tilde{V}\big)\setminus \{\, y_n = 0 \,\}\Big)$$

and, for all $\varphi \in H^1_0(\tilde{V})$, v solves the equation

$$\int_{\{y_n<\varepsilon\}\cap\tilde{V}} \varepsilon(y_n) A_\varepsilon \nabla v\cdot\nabla\varphi\, dy + \beta\int_{\{y_n=\varepsilon\}\cap\tilde{V}} v\varphi J_\varepsilon\, d\mathcal{H}^{n-1} = \int_{\{y_n<0\}\cap\tilde{V}} \tilde{f}_\varepsilon\varphi\, dy, \tag{4.77}$$

where

$$\varepsilon(y_n) = \begin{cases} \varepsilon & \text{if } y_n > 0, \\ 1 & \text{if } y_n \le 0, \end{cases}$$

$$A_\varepsilon(y) = k_\varepsilon(y')(D(\Phi^{-1})(y))^{-1}(D(\Phi^{-1})(y))^{-T}, \qquad \tilde{f}_\varepsilon(y) = f(\Phi^{-1}(y))\, k_\varepsilon(y'),$$

and

$$J_\varepsilon(y) = \sqrt{1+|\nabla g(y') + \varepsilon\nabla k_\varepsilon(y')|^2}.$$

Notice that A_ε is elliptic and bounded, uniformly in y and ε. Moreover, using (4.66) in Corollary 4.44, we also get that there exists a positive constant $C = C(\Omega, h, \beta, f, \sigma_0)$ such that

$$\int_{\{\, y_n<\varepsilon \,\}\cap\tilde{V}} \varepsilon(y_n)|\nabla v|^2\, dy + \beta\int_{\{y_n=\varepsilon\}\cap\tilde{V}} v^2\, d\mathcal{H}^{n-1} \le C. \tag{4.78}$$

H^2 uniform estimates

Since we aim to prove H^2 estimates with a local approach, we define the energy quantities I_δ and $\tilde{I}_\delta$ as follows: given a function

$$\varphi \in H^1\big(\{\, y_n < \varepsilon \,\}\cap \tilde{V}\big) \cap H^2\Big(\big(\{\, y_n < \varepsilon \,\}\cap \tilde{V}\big)\setminus \{\, y_n = 0 \,\}\Big)$$

and $0 < \delta < \delta_0$, we denote by

$$\tilde{I}_{\delta,\sigma_0}(\varphi) = \int_{\{\, y_n<\varepsilon \,\}\cap\tilde{Q}_\delta} \varepsilon(y_n)|D^2\varphi|^2\, dy + \beta\int_{\{y_n=\varepsilon\}\cap\tilde{Q}_\delta} |\nabla_{n-1}\varphi|^2\, d\mathcal{H}^{n-1}, \tag{4.79}$$

where

$$\nabla_{n-1}\varphi = \nabla\varphi - \frac{\partial\varphi}{\partial y_n}\mathbf{e}_n,$$

and, as in Remark 4.47,

$$\varepsilon(t) = \begin{cases} \varepsilon & \text{if } t > 0, \\ 1 & \text{if } t \le 0. \end{cases}$$

Analogously in Ω_ε, for every

$$\varphi \in H^1\big(\Omega_\varepsilon \cap V\big) \cap H^2\Big(\big(\Omega_\varepsilon \cap V\big)\backslash\partial\Omega\Big),$$

we let

$$I_{\delta,\sigma_0}(\varphi) = \int_{\Omega_\varepsilon\cap Q_\delta} \varepsilon(d(x))|D^2\varphi|^2\,dx + \beta\int_{\partial\Omega_\varepsilon\cap Q_\delta} |\nabla^{\partial\Omega_\varepsilon}\varphi|^2\,d\mathcal{H}^{n-1}, \tag{4.80}$$

where we recall that

$$\nabla^{\partial\Omega_\varepsilon}\varphi = \nabla\varphi - (\nabla\varphi\cdot\nu_\varepsilon)\nu_\varepsilon.$$

When possible, we will drop the dependence on σ_0. Uniform bounds for I_δ can be read as uniform bounds for $\tilde{I}_\delta$ and vice versa. Indeed, we have the following

Lemma 4.48. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $\varepsilon\|h\|_{C^{0,1}} \le \varepsilon_0$ and $h(x) = h(\sigma(x))$. Then, for every $\sigma_0 \in \partial\Omega$, there exists a positive constant $C = C(\Omega, \|h\|_{C^{1,1}}, \sigma_0)$ such that if*

$$u \in H^1(Q_\delta) \cap H^2(Q_\delta \setminus \partial\Omega),$$

and

$$v(y) = u(\Phi^{-1}(y)),$$

then, for every $0 < \delta < \delta_0$,

$$I_\delta(u) \le C\left(\tilde{I}_\delta(v) + \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_\delta} \varepsilon(y_n)|\nabla v|^2\,dy\right), \tag{4.81}$$

and

$$\tilde{I}_\delta(v) \le C\left(I_\delta(u) + \int_{\Omega_\varepsilon\cap Q_\delta} \varepsilon(d(x))|\nabla u|^2\,dx\right). \tag{4.82}$$

Proof. We start by evaluating the trace term in the definition of $\tilde{I}_\delta$. By means of the change of variables $y = \Phi(x)$, we have

$$\int_{\{y_n=\varepsilon\}\cap\tilde{Q}_\delta} |\nabla_{n-1}v|^2\,d\mathcal{H}^{n-1}(y) = \sum_{i=1}^{n-1}\int_{\partial\Omega_\varepsilon\cap V} (\nabla u\cdot w_i)^2\,J^{\partial\Omega_\varepsilon}\Phi\,d\mathcal{H}^{n-1}(x), \tag{4.83}$$

where

$$w_i = \mathbf{e}_i + (\partial_i g + \varepsilon\partial_i k_\varepsilon)\,\mathbf{e}_n.$$

The vectors w_i (that, in general, could be non-orthogonal) form a basis for the tangent plane $T_{\sigma_0}\partial\Omega_\varepsilon$. In particular, we have that there exists a positive constant $C = C(\Omega, h, \sigma_0)$ such that

$$|\nabla^{\partial\Omega_\varepsilon}u|^2 \le C\sum_{i=1}^{n-1}(\nabla u\cdot w_i)^2 \le C^2|\nabla^{\partial\Omega_\varepsilon}u|^2. \tag{4.84}$$

Therefore, using the uniform bounds (4.73) we get that for some positive constant $C = C(\Omega, h)$

$$\frac{1}{C} \le J^{\partial\Omega_\varepsilon}\Phi = \frac{1}{\sqrt{1 + |\nabla g + \varepsilon\nabla k_\varepsilon|^2}} \le C, \tag{4.85}$$

and joining (4.83), (4.84), and (4.85), we get

$$\int_{\partial\Omega_\varepsilon\cap Q_\delta}|\nabla^{\partial\Omega_\varepsilon}\varphi|^2\,d\mathcal{H}^{n-1}\le C\int_{\{y_n=\varepsilon\}\cap\tilde{Q}_\delta}|\nabla_{n-1}v|^2\,d\mathcal{H}^{n-1}\le C^2\int_{\partial\Omega_\varepsilon\cap Q_\delta}|\nabla^{\partial\Omega_\varepsilon}\varphi|^2\,d\mathcal{H}^{n-1}. \quad (4.86)$$

For what concerns the second order term, we evaluate for a.e. $y\in\{y_n<\varepsilon\}\cap\tilde{V}$,

$$\big(D^2v(y)\big)_{ij}=\sum_{k=1}^{n}\frac{\partial^2(\Phi^{-1})_k}{\partial y_i\,\partial y_j}\frac{\partial u}{\partial x_k}+\sum_{k,l=1}^{n}\frac{\partial(\Phi^{-1})_k}{\partial y_i}\frac{\partial^2 u}{\partial x_k\,\partial x_l}\frac{\partial(\Phi^{-1})_l}{\partial y_j},$$

so that, for some positive constant $C=C(n,\|\Phi^{-1}\|_{C^{1,1}})$,

$$|D^2v\big(\Phi(x)\big)|^2\le C\big(|\nabla u(x)|^2+|D^2u(x)|^2\big). \quad (4.87)$$

Similarly we have that, for a.e. $x\in\Omega_\varepsilon\cap V$, and for some positive constant $C=C(n,\|\Phi\|_{C^{1,1}})$,

$$|D^2u\big(\Phi^{-1}(y)\big)|^2\le C\big(|\nabla v(y)|^2+|D^2v(y)|^2\big). \quad (4.88)$$

Using (4.87) and the uniform bounds on k_ε, we get

$$\begin{aligned}\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_\delta}\varepsilon(y_n)|D^2v|^2\,dy&=\int_{\Omega_\varepsilon\cap Q_\delta}\frac{\varepsilon(d(x))}{k_\varepsilon}|D^2v\big(\Phi(x)\big)|^2\,dx\\&\le C\int_{\Omega_\varepsilon\cap Q_\delta}\varepsilon(d(x))\big(|D^2u|^2+|\nabla u|^2\big)\,d\mathcal{H}^{n-1}(x).\end{aligned} \quad (4.89)$$

Analogously, using (4.88),

$$\int_{\Omega_\varepsilon\cap Q_\delta}\varepsilon(d(x))|D^2u|^2\,dx\le C\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_\delta}\varepsilon(y_n)\big(|D^2v|^2+|\nabla v|^2\big)\,d\mathcal{H}^{n-1}. \quad (4.90)$$

The result now follows by joining (4.86), (4.90), and (4.89). $\square$

Lemma 4.49. *Let $\Omega\subset\mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, fix a positive function $h\in C^{1,1}(\Gamma_{d_0})$ such that $\varepsilon\|h\|_{C^{0,1}}\le\varepsilon_0$ and $h(x)=h(\sigma(x))$. If $\sigma_0\in\partial\Omega$, and v is as in Remark 4.47, then*

$$v\in H^2\Big(\{\,y_n<\varepsilon\,\}\cap\tilde{Q}_{\delta_0/2}\setminus\{\,y_n=0\,\}\Big),$$

and there exists a positive constant $C=C(\Omega,\|h\|_{C^{1,1}},\beta,f,\sigma_0)$ such that

$$\tilde{I}_{\delta_0/2}(v)\le C. \quad (4.91)$$

Proof. Let $\xi\in C_c^\infty(\tilde{Q}_{\delta_0})$ be a non-negative function with $\xi\equiv1$ in $\tilde{Q}_{\delta_0/2}$. For $|\eta|$ small enough, we have

$$\operatorname{supp}\xi+\eta\mathbf{e}_i\subset\subset\tilde{Q}_{\delta_0}$$

for every $k=1,\dots,n-1$. In the following, for any L^2 function ψ, we denote by

$$\Delta_k^\eta\psi(x)=\frac{\psi(x+\eta\mathbf{e}_k)-\psi(x)}{\eta}$$

the difference quotients (see for instance [39, §5.8.2]). We recall that for any couple of functions ψ_1 and ψ_2 we have

$$\Delta_k^\eta(\psi_1\psi_2)(x)=\Delta_k^\eta\psi_1(x)\psi_2(x)+\psi_1(x+\eta\mathbf{e}_k)\Delta_k^\eta\psi_2(x),$$

and if ψ_1 and ψ_2 are measurable and

$$(\operatorname{supp}\psi_1 \cap \operatorname{supp}\psi_2) \pm \eta \mathbf{e}_k \subset\subset \tilde{Q}_\delta,$$

then for every $k = 1, \dots, n-1$, it holds

$$\int_{\tilde{Q}_{\delta_0}} \psi_1(y) \Delta_k^\eta \psi_2(y)\, dy = - \int_{\tilde{Q}_{\delta_0}} \Delta_k^{-\eta} \psi_1(y)\, \psi_2(y)\, dy. \tag{4.92}$$

Moreover, we recall that if $\psi \in H^1(\{y_n < \varepsilon\} \cap \tilde{Q}_{\delta_0})$, then for every $U \subset\subset \tilde{Q}_{\delta_0}$ we have that

$$\int_U (\Delta_k^\eta \psi)^2\, dy \le \int_{\tilde{Q}_{\delta_0}} |\nabla \psi|^2\, dy, \tag{4.93}$$

for every $|\eta| < d(U, \partial \tilde{Q}_{\delta_0})$. We aim to prove some uniform L^2 estimates for $\Delta_k^\eta(\nabla v)$, which will imply weak differentiability and uniform L^2 estimates for $\partial_k \nabla v$.

Given the equation (4.77), we use it as a test function

$$\varphi = -\Delta_k^{-\eta}(\xi^2 \Delta_k^\eta v).$$

For small $|\eta|$, the function φ is admissible. Here, we recall the weak equation

$$\underbrace{\int_{\{y_n<\varepsilon\}\cap \tilde{V}} \varepsilon(y_n) A_\varepsilon \nabla v \cdot \nabla \varphi\, dy}_{\mathcal{I}_1} + \underbrace{\beta \int_{\{y_n=\varepsilon\}\cap \tilde{V}} v \varphi J_\varepsilon\, d\mathcal{H}^{n-1}}_{\mathcal{I}_2} = \underbrace{\int_{\{y_n<0\}\cap \tilde{V}} \tilde{f}_\varepsilon \varphi\, dy}_{\mathcal{I}_3}.$$

We now estimate the integrals separately. In the following, we use the Einstein notation on repeated indices, where $i, j = 1, \dots, n$, and for simplicity we drop the dependence on ε in A_ε, whose components are denoted by a_{ij}. We recall that $a_{ij} = a_{ij}(y)$ and $J_\varepsilon = J_\varepsilon(y)$. When necessary, we will also write $a_{ij}^\eta := a_{ij}(y + \eta \mathbf{e}_k)$ and $J_\varepsilon^\eta := J_\varepsilon(y + \eta \mathbf{e_k})$.

Let us start with the higher-order term $\mathcal{I}_1$: by direct computation, we get

$$\begin{aligned} A \nabla v \cdot \nabla \varphi &= -a_{ij}\, \partial_j v\, \Delta_k^{-\eta}\, \partial_i(\xi^2 \Delta_k^\eta v) \\ &= -a_{ij}\, \partial_j v \left[\Delta_k^{-\eta}(\partial_i(\xi^2)\, \Delta_k^\eta v) + \Delta_k^{-\eta}(\xi^2 \Delta_k^\eta(\partial_i v)) \right] \end{aligned}$$

Multiplying by $\varepsilon(y_n)$, integrating over $\{y_n < \varepsilon\} \cap \tilde{Q}_{\delta_0}$, and using property (4.92), we get (notice that $\varepsilon(y_n)$ is constant along the direction $\mathbf{e}_k$)

$$\begin{aligned} \mathcal{I}_1 &= \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n) \Delta_k^\eta(a_{ij}\, \partial_j v)\, \partial_i(\xi^2) \Delta_k^\eta v\, dy \\ &\quad + \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n) \Delta_k^\eta(a_{ij} \partial_j v)\, \xi^2\, \Delta_k^\eta(\partial_i v)\, dy \\ &= \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n) \xi^2\, a_{ij}^\eta\, \Delta_k^\eta(\partial_j v)\, \Delta_k^\eta(\partial_i v)\, dy + R_1 \\ &\ge C_1 \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n) \xi^2 \sum_{j=1}^n \left(\Delta_k^\eta(\partial_j v) \right)^2 dy - |R_1|, \end{aligned} \tag{4.94}$$

where C_1 is the ellipticity constant of A estimated uniformly in Remark 4.47, and

$$\begin{aligned} R_1 = 2 &\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n) \Big(\Delta_k^\eta(a_{ij}) \partial_j v + a_{ij}^\eta\, \Delta_k^\eta(\partial_j v) \Big)\, \xi\, \partial_i \xi\, \Delta_k^\eta v\, dy \\ &+ \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n) \Delta_k^\eta(a_{ij}) \partial_j v\, \xi^2\, \Delta_k^\eta(\partial_j v)\, dy. \end{aligned}$$

Using Young's inequality

$$ab \leq \frac{\lambda}{2}a^2 + \frac{1}{2\lambda}b^2 \qquad \forall a, b \geq 0,$$

with a suitable choice of $\lambda > 0$, and using the uniform bounds on $\|\xi\|_{C^1}$ and $\|A_\varepsilon\|_{C^1}$ (see Remark 4.47 and Remark 4.46), we may obtain a positive constant $C = C(A, \xi)$ such that the following estimate holds

$$\begin{aligned}|R_1| \leq \frac{C_1}{2} &\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\xi^2 \sum_{j=1}^n \left(\Delta_k^\eta(\partial_j v)\right)^2 dy \\ &+ C\int_{\{y_n<\varepsilon\}\cap \operatorname{supp}\xi} \varepsilon(y_n)\left((\Delta_k^\eta v)^2 + \sum_{j=1}^n (\partial_j v)^2\right) dy.\end{aligned}$$

This estimate, joint with (4.94), implies

$$\mathcal{I}_1 \geq \frac{C_1}{2}\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\xi^2 \sum_{j=1}^n \left(\Delta_k^\eta(\partial_j v)\right)^2 dy - C\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)|\nabla v|^2\, dy. \tag{4.95}$$

Similarly, let us work with the boundary terms $\mathcal{I}_2$, where we have

$$v\varphi J_\varepsilon = -vJ_\varepsilon\Delta_k^{-\eta}(\xi^2\Delta_k^\eta v).$$

Since $k \neq n$, we can use an analogous of property (4.92) also on $\{\, y_n = \varepsilon \,\} \cap \tilde{Q}_{\delta_0}$, so that

$$\begin{aligned}\mathcal{I}_2 &= \beta\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} \xi^2\Delta_k^\eta(vJ_\varepsilon)\Delta_k^\eta v\, d\mathcal{H}^{n-1} \\ &= \beta\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} \xi^2(\Delta_k^\eta v)^2 J_\varepsilon^\eta\, d\mathcal{H}^{n-1} + R_2 \\ &\geq \beta C_2\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} \xi^2(\Delta_k^\eta v)^2\, d\mathcal{H}^{n-1} - |R_2|,\end{aligned} \tag{4.96}$$

where $C_2 = \inf_{Q_\delta} J_\varepsilon$ uniformly estimated in Remark 4.47, and

$$R_2 = \beta\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} \xi^2\Delta_k^\eta(J_\varepsilon)v\Delta_k^\eta v\, d\mathcal{H}^{n-1}$$

As done for $\mathcal{I}_1$, using Young's inequality and the uniform bounds on $\|J_\varepsilon\|_{C^1}$ (see Remark 4.47 and Remark 4.46), we have that for some positive constant $C = C(J_\varepsilon, \xi)$,

$$|R_2| \leq \frac{\beta C_2}{2}\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} \xi^2(\Delta_k^\eta v)^2\, d\mathcal{H}^{n-1} + C\beta\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} v^2\, d\mathcal{H}^{n-1}$$

This estimate, joint with (4.96) ensures

$$\mathcal{I}_2 \geq \frac{\beta C_2}{2}\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} \xi^2(\Delta_k^\eta v)^2\, d\mathcal{H}^{n-1} - C\beta\int_{\{\, y_n=\varepsilon\,\}\cap\tilde{Q}_{\delta_0}} v^2\, d\mathcal{H}^{n-1}. \tag{4.97}$$

Finally, we estimate the source term $\mathcal{I}_3$, recalling that

$$\tilde{f}_\varepsilon\varphi = -\tilde{f}_\varepsilon\Delta_k^{-\eta}(\xi^2\Delta_k^\eta v).$$

Using Young's inequality and property (4.93) for $\psi = \xi^2 \Delta_k^\eta v$ (notice that $\nabla \Delta_k^\eta v = \Delta_k^\eta (\nabla v)$), then for a suitable positive constant $C = C(\tilde{f}, \xi, \delta_0)$,

$$\begin{aligned}
\mathcal{I}_3 \leq \frac{C_1}{4} &\int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} \xi^2 \sum_{j=1}^n (\Delta_k^\eta \partial_j v)^2 \, dy \\
&+ C \int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy + C \int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} |\nabla v|^2 \, dy \\
\leq \frac{C_1}{4} &\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0}} \varepsilon(y_n) \xi^2 \sum_{j=1}^n (\Delta_k^\eta \partial_j v)^2 \, dy \\
&+ C \int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy + C \int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0}} \varepsilon(y_n) |\nabla v|^2 \, dy.
\end{aligned} \tag{4.98}$$

We can now turn back to the equation

$$\mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I}_3$$

Joining (4.95), (4.97), (4.98), and using the fact that $\xi = 1$ in $\tilde{Q}_{\delta_0/2}$, then, for every $k = 1, \dots, n-1$,

$$\begin{aligned}
\mathcal{I}_4 := \frac{C_1}{4} &\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0/2}} \varepsilon(y_n) \sum_{j=1}^n (\Delta_k^\eta \partial_j v)^2 \, dy \\
&+ \frac{\beta C_2}{2} \int_{\{ y_n = \varepsilon \} \cap \tilde{Q}_{\delta_0/2}} (\Delta_k^\eta v)^2 \, d\mathcal{H}^{n-1} \\
\leq C \Bigg(&\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0}} \varepsilon(y_n) |\nabla v|^2 \, dy \\
&+ \beta \int_{\{ y_n = \varepsilon \} \cap \tilde{Q}_{\delta_0}} v^2 \, d\mathcal{H}^{n-1} + \int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy \Bigg).
\end{aligned} \tag{4.99}$$

Since we have assumed the uniform H^1 estimates (4.66), and we have uniform estimates for $\tilde{f}_\varepsilon$, computed in Remark 4.47, we get that for some positive constant $C = C(\delta_0, \Omega, \xi)$

$$\mathcal{I}_4 \leq C,$$

which implies that for every $k = 1, \dots, n-1$, we have $\partial_k v \in H^1(\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0/2})$ and

$$\begin{aligned}
\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0/2}} \varepsilon(y_n) \sum_{j=1}^n (\partial_k \partial_j v)^2 \, dy \leq C \Bigg(&\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0}} \varepsilon(y_n) |\nabla v|^2 \, dy \\
&+ \beta \int_{\{ y_n = \varepsilon \} \cap \tilde{Q}_{\delta_0}} v^2 \, d\mathcal{H}^{n-1} + \int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy \Bigg).
\end{aligned}$$

It only remains to get a uniform estimate for $\partial_{nn}^2 v$. We notice that since v solves (4.77), then almost everywhere we have

$$- \sum_{\substack{1 \leq i,j \leq n \\ (i,j) \neq (n,n)}} \varepsilon(y_n) \partial_i (a_{ij} \partial_j v) - \varepsilon(y_n) \partial_n a_{nn} \partial_n v - \varepsilon(y_n) a_{nn} \partial_{nn}^2 v = \tilde{f} \chi_{\{ y_n < 0 \}}.$$

In particular, the fact that $a_{nn} = k_\varepsilon$ is uniformly bounded from below gives estimates for $\partial^2_{nn} v$ in terms of A_ε, $\tilde{f}$, and the other derivatives of v. Therefore, so that, for every $0 < \delta < \delta_0$, we can find a positive constant $C = C(\Omega, h, f)$ such that

$$\begin{aligned}\int_{\{\, y_n<\varepsilon\,\}\cap\tilde{Q}_\delta} \varepsilon(y_n)|\partial^2_{nn} v|^2 \le C\Bigg(\int_{\{\, y_n<\varepsilon\,\}\cap\tilde{Q}_\delta} \varepsilon(y_n)\sum_{k=1}^{n-1}|\nabla\partial_k v|^2\,dy\\ + \int_{\{\, y_n<\varepsilon\,\}\cap\tilde{Q}_\delta} \varepsilon(y_n)|\nabla v|^2\,d\mathcal{H}^{n-1} + 1\Bigg).\end{aligned} \tag{4.100}$$

Finally, (4.99), joint with (4.100) and the bound (4.78), gives (4.91).

□

Remark 4.50. Let $\sigma_0 \in \partial\Omega$, by the previous lemma we have that $u_\varepsilon \in H^2(Q_{\delta_0/2}\setminus\partial\Omega)$. Moreover, putting together estimates (4.81) and (4.82) and the bounds on the first derivative ((4.66) and (4.78)), we have

$$I_\delta(u_\varepsilon) \le C(1 + \tilde{I}_\delta(v_\varepsilon)) \le C^2(1 + I_\delta(u_\varepsilon)). \tag{4.101}$$

We can finally prove Theorem 4.42.

Proof of Theorem 4.42. We recall that we have defined $\tilde{I}_{\delta,\sigma}$ and $I_{\delta,\sigma}$ in (4.79) and (4.80) respectively. For every $\sigma_0 \in \partial\Omega$, using (4.91) from Lemma 4.49, and (4.101) from Remark 4.50, we have that there exists a constant $C = C(\Omega, h, \sigma_0, \delta_0)$ such that

$$I_{\delta_0/2,\sigma_0}(u_\varepsilon) \le C. \tag{4.102}$$

From the boundedness of Ω there exist $\sigma_1, \dots, \sigma_m \in \partial\Omega$ and associated $\delta_i = \delta(\sigma_i, \Omega)$ for which estimates of the type (4.102) hold and such that, choosing $\varepsilon_0 = \min\{\varepsilon_0(\sigma_1), \dots, \varepsilon_0(\sigma_m)\} > 0$, we have

$$\Sigma_\varepsilon \subset \bigcup_{i=1}^m \Phi^{-1}_{\sigma_i}(\tilde{Q}_{\delta_i/2}) = V_0.$$

Let U be an open set such that $\overline{U} \subset \Omega$ and $\Omega_\varepsilon \subset U \cup V_0$ for every ε such that $\varepsilon\|h\|_{C^{0,1}} < \varepsilon_0$. Also in U we may get an estimate analogous to (4.102). Indeed, by standard elliptic regularity and by estimate (4.67), we have that

$$\int_U |D^2 u_\varepsilon|^2\,dx \le C\bigg(\int_\Omega f^2\,dx + \int_\Omega u_\varepsilon^2\bigg) \le C. \tag{4.103}$$

Let

$$I(u_\varepsilon) = \int_\Omega |D^2 u_\varepsilon|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon}|D^2 u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon}|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2\,d\mathcal{H}^{n-1},$$

summing from $i = 1$ to m the estimates of the type (4.102) and (4.103), we have that there exists $C = C(\Omega, h, f)$ such that

$$I(u_\varepsilon) \le C,$$

and the assertion is proven. □

For every $x \in \Gamma_{d_0}$, we denote by

$$\nabla^{\partial\Omega} u_\varepsilon(x) = \nabla u_\varepsilon(x) - (\nabla u_\varepsilon(x)\cdot\nu_0(x))\nu_0(x),$$

and we have the following.

Corollary 4.51. *There exists a positive constant $C = C(\Omega, h, \beta, f)$ such that*

$$\int_{\Sigma_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2 \, dx \le \varepsilon C.$$

Proof. We start by proving that on $\partial\Omega_\varepsilon$

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 \ge (1-\varepsilon^2)|\nabla^{\partial\Omega} u_\varepsilon|^2. \tag{4.104}$$

Indeed, for every $\xi \in \partial\Omega_\varepsilon$, if $|\nabla h(\xi)| = 0$, then $\nu_0 = \nu_\varepsilon$ and the inequality is trivially true; on the other hand, if $|\nabla h(\xi)| \neq 0$ we have that

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 = |\nabla u_\varepsilon|^2 - |\nabla u_\varepsilon \cdot \nu_\varepsilon|^2,$$

and for every $\eta > 0$

$$|\nabla u_\varepsilon \cdot \nu_\varepsilon|^2 \le \frac{(1+\eta)|\nabla u_\varepsilon \cdot \nu_0|^2 + \left(1+\dfrac{1}{\eta}\right)\varepsilon^2 |\nabla u_\varepsilon \cdot \nabla h|^2}{1+|\nabla h|^2}.$$

Moreover $\nabla h \cdot \nu_0 = 0$, so that

$$|\nabla u_\varepsilon \cdot \nabla h|^2 = |\nabla^{\partial\Omega} u_\varepsilon \cdot \nabla h|^2 \le (|\nabla u_\varepsilon|^2 - |\nabla u_\varepsilon \cdot \nu_0|^2)|\nabla h|^2.$$

Therefore, for every $\eta > 0$

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 \ge \left(1 - \frac{\left(1+\dfrac{1}{\eta}\right)\varepsilon^2|\nabla h|^2}{1+|\nabla h|^2}\right)|\nabla u_\varepsilon|^2 - \frac{1+\eta-\left(1+\dfrac{1}{\eta}\right)\varepsilon^2|\nabla h|^2}{1+|\nabla h|^2}|\nabla u_\varepsilon \cdot \nu_0|^2,$$

finally, letting $\eta = |\nabla h|^2$, we have

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 \ge (1-\varepsilon^2)(|\nabla u_\varepsilon|^2 - |\nabla u_\varepsilon \cdot \nu_0|^2) = (1-\varepsilon^2)|\nabla^{\partial\Omega} u_\varepsilon|^2.$$

Then, by Theorem 4.42 and (4.104), we have that

$$\varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2 \, d\mathcal{H}^{n-1} \le C. \tag{4.105}$$

For $x \in \Sigma_\varepsilon$ denote by

$$\xi(x) = \sigma(x) + \varepsilon h(x)\nu_0(x) \in \partial\Omega_\varepsilon,$$

so that for $\mathcal{L}^n$-a.e. $x \in \Sigma_\varepsilon$, we have that

$$\nabla^{\partial\Omega} u_\varepsilon(x) = \nabla^{\partial\Omega} u_\varepsilon(\xi(x)) - \int_{d(x)}^{\varepsilon h(x)} \frac{d}{dt}\Big(\nabla^{\partial\Omega} u_\varepsilon(\sigma(x) + t\nu_0(x))\Big)\, dt.$$

Hence,

$$|\nabla^{\partial\Omega} u_\varepsilon|(x)^2 \le C(h)\left(|\nabla^{\partial\Omega} u_\varepsilon|^2(\xi(x)) + \varepsilon \int_0^{\varepsilon h(x)} |D^2 u_\varepsilon|^2(\sigma(x) + t\nu_0(x))\, dt\right),$$

and integrating over Σ_ε, using (3.45) and (3.46), and (4.105), we get

$$\begin{aligned}\int_{\Sigma_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2(x)\,dx \le C \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} |\nabla^{\partial\Omega} u_\varepsilon|^2(\sigma + \varepsilon h(\sigma)\nu_0)\,ds\,d\mathcal{H}^{n-1}(\sigma) + \\ + C\varepsilon \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)}\int_0^{\varepsilon h(\sigma)} |D^2 u_\varepsilon|^2(\sigma(x) + t\nu_0(x))\,dt\,ds\,d\mathcal{H}^{n-1}(\sigma) \\ \le \varepsilon C\Big(\int_{\partial\Omega_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2\,d\mathcal{H}^{n-1} + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^2\,dx\Big) \\ \le \varepsilon C.\end{aligned}$$

□

4.3.3 First-order development

We can now prove Theorem 4.35

Proof of Theorem 4.35. We recall that we are assuming without loss of generality $\|h\|_\infty < d_0$, so that $d \in C^{1,1}(\Sigma_1)$. To prove the equiboundedness in $H^1(\Omega_1)$, we decompose $\nabla\tilde{u}_\varepsilon$ into its normal part

$$(\nabla\tilde{u}_\varepsilon \cdot \nu_0)\,\nu_0,$$

and its tangential part

$$\nabla^{\partial\Omega}\tilde{u}_\varepsilon := \nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \nu_0)\nu_0.$$

Using Lemma 4.38, since J_ε and J_ε^τ are equibounded, we can find a positive constant $C = C(\Omega, h)$ such that

$$\int_{\Sigma_1} |\nabla^{\partial\Omega}\tilde{u}_\varepsilon|^2\,dz \le \frac{C}{\varepsilon}\int_{\Sigma_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2\,dx, \tag{4.106}$$

$$\int_{\Sigma_1} |\nabla\tilde{u}_\varepsilon \cdot \nu_0|^2\,dz \le \varepsilon C \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx, \tag{4.107}$$

and

$$\int_{\partial\Omega_1} \tilde{u}_\varepsilon^2\,d\mathcal{H}^{n-1} \le C \int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1}. \tag{4.108}$$

Moreover, by the weak convergence of u_ε in $H^1(\Omega)$, u_ε are equibounded in $L^2(\Omega)$, while by the minimality

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} &\le \mathcal{E}_{p,\varepsilon}(0) + 2\int_\Omega f u_\varepsilon\,dx \\ &\le \int_\Omega (f^2 + u_\varepsilon^2)\,dx \\ &\le C\end{aligned} \tag{4.109}$$

Joining the inequalities (4.106), (4.107), (4.108), and (4.109) with the energy estimates in Corollary 4.51, we have that for some positive constant $C(\Omega, h, \beta)$

$$\int_{\Omega_1} |\nabla\tilde{u}_\varepsilon|^2\,dz + \beta\int_{\partial\Omega_1} \tilde{u}_\varepsilon^2\,d\mathcal{H}^{n-1} \le C.$$

By Poincaré's inequality with boundary term, we have that $\tilde{u}_\varepsilon$ are equibounded in $H^1(\Omega_1)$. Therefore, up to a subsequence, $\tilde{u}_\varepsilon$ converges weakly in $H^1(\Omega_1)$ to some function $\tilde{u}_0$. In particular, by the weak convergence in $H^1(\Omega)$ of u_ε to u_0, we have $\tilde{u}_0 = u_0$ in Ω. On the other hand, in Σ_1, using Proposition 4.39, we get that $\tilde{u}_0$ is a solution to (4.58), so that Remark 4.41 ensures that

$$\tilde{u}_0(z) = \tilde{u}_0(\sigma(z))\left(1 - \frac{\beta d(z)}{1+\beta h(z)}\right).$$

Finally, since $\tilde{u}_0 \in H^1(\Omega_1)$, then it cannot jump across $\partial\Omega$, so that, since $\tilde{u}_0 = u_0$ in Ω, we necessarily have that $\tilde{u}_0(\sigma(z)) = u_0(\sigma(z))$ for a.e. $z \in \Sigma_1$, and the theorem is proved. $\square$

Following the approach of Theorem 3.15 we can use Theorem 4.35 to prove Theorem 4.36

Proof of Theorem 4.36. We start by proving the Γ-liminf inequality. Without loss of generality, we can prove the inequality for the sequence of minimisers u_ε. Here we recall the definitions of $\mathcal{E}_{p,\varepsilon}$ and $\mathcal{E}_{p,0}$, omitting the dependence on h.

$$\mathcal{E}_{p,\varepsilon}(u_\varepsilon) = \int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} - 2\int_\Omega f u_\varepsilon\,dx, \tag{4.110}$$

$$\mathcal{E}_{p,0}(u_0) = \int_\Omega |\nabla u_0|^2\,dx + \beta\int_{\partial\Omega} \frac{u_0^2}{1+\beta h}\,d\mathcal{H}^{n-1} - 2\int_\Omega f u_0\,dx. \tag{4.111}$$

Moreover, notice that, by minimality of u_0,

$$\mathcal{E}_{p,\varepsilon}(u_\varepsilon) - \mathcal{E}_{p,0}(u_0) \geq \varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} - \beta\int_{\partial\Omega}\frac{u_\varepsilon^2}{1+\beta h}\,d\mathcal{H}^{n-1}. \tag{4.112}$$

Arguing exactly as in the proof of Theorem 3.15, we have

$$\begin{aligned}\delta\mathcal{E}_{p,\varepsilon}(u_\varepsilon) =& \frac{\mathcal{E}_{p,\varepsilon}(u_\varepsilon) - \mathcal{E}_{p,0}(u_0)}{\varepsilon} \geq \int_{\partial\Omega}\frac{1}{\varepsilon h(\sigma)}(1-\lambda+\beta h)(1+\varepsilon h H)u_\varepsilon^2(\sigma+\varepsilon h\nu_0)\,d\mathcal{H}^{n-1}\\ &+\int_{\partial\Omega}\frac{1}{\varepsilon h}\left(\left(1-\frac{1}{\lambda}\right)\left(u_\varepsilon(\sigma)+\int_0^{\varepsilon h}\frac{Hu_\varepsilon(\sigma+t\nu_0)}{2\sqrt{1+tH}}\,dt\right)^2 - \frac{\beta h\,u_\varepsilon^2(\sigma)}{1+\beta h}\right)d\mathcal{H}^{n-1}\\ &- Q\varepsilon R(\varepsilon,u_\varepsilon)\end{aligned} \tag{4.113}$$

where, if ε is small enough,

$$\begin{aligned}R(\varepsilon,u_\varepsilon) &= \varepsilon\int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)}|\nabla u_\varepsilon(\sigma+t\nu_0)|^2\,d\mathcal{H}^{n-1} + \beta\int_{\partial\Omega}u_\varepsilon(\sigma+\varepsilon h(\sigma)\nu_0(\sigma))^2\,d\mathcal{H}^{n-1}\\ &\leq C\int_\Omega f u_\varepsilon\,dx < C.\end{aligned}$$

Letting $\lambda = \lambda(\sigma) = 1+\beta h(\sigma)$ in (4.113), and using the inequality $(a+b)^2 \geq a^2+2ab$, joint with the fact that $1-\lambda^{-1}>0$,

$$\delta\mathcal{E}_{p,\varepsilon}(u_\varepsilon) \geq \int_{\partial\Omega}\frac{\beta H u_\varepsilon(\sigma)}{\varepsilon(1+\beta h)}\int_0^{\varepsilon h(\sigma)}\frac{u_\varepsilon(\sigma+t\nu_0)}{\sqrt{1+tH}}\,dt\,d\mathcal{H}^{n-1} + O(\varepsilon).$$

Moreover, for every $t \in (0,\varepsilon\|h\|_\infty)$ we have that $(1+tH)^{-1/2} = 1+O(\varepsilon)$, so that

$$\delta\mathcal{E}_{p,\varepsilon}(u_\varepsilon) \geq \beta\int_{\partial\Omega}\frac{Hu_\varepsilon(\sigma)}{(1+\beta h)}\int_0^{h(\sigma)}\tilde{u}_\varepsilon(\sigma+t\nu_0)\,dt\,d\mathcal{H}^{n-1} + O(\varepsilon). \tag{4.114}$$

Finally, by Theorem 4.35 we get

$$\tilde{u}_\varepsilon \xrightarrow{L^2(\Sigma_1)} \tilde{u}_0,$$

so that

$$\int_0^{h(\sigma)} \tilde{u}_\varepsilon(\sigma + t\nu_0)\, dt \xrightarrow{L^2(\partial\Omega)} \int_0^{h(\sigma)} \tilde{u}_0(\sigma + t\nu_0)\, dt.$$

Indeed, by (3.45),

$$\int_{\partial\Omega} \left(\int_0^{h(\sigma)} (\tilde{u}_\varepsilon(\sigma + t\nu_0) - \tilde{u}_0(\sigma + t\nu_0))\, dt \right)^2 d\mathcal{H}^{n-1} \leq C \int_{\Sigma_1} (\tilde{u}_\varepsilon - \tilde{u}_0)^2\, dz.$$

Therefore, passing to the limit in (4.114), and using the explicit expression of $\tilde{u}_0$, we get

$$\liminf_{\varepsilon \to 0^+} \delta\mathcal{E}_{p,\varepsilon}(u_\varepsilon) \geq \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2} u_0^2(\sigma)\, d\mathcal{H}^{n-1},$$

and the Γ-liminf is proved.

We now prove the Γ-limsup inequality.

Let

$$\varphi_\varepsilon(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ u_0(\sigma(x)) \left(1 - \dfrac{\beta d(x)}{\varepsilon(1+\beta h(x))} \right) & \text{if } x \in \Sigma_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases}$$

where we recall that $\nabla h \cdot \nu_0 = 0$. We have that $\varphi_\varepsilon \in H^1(\Omega)$ and φ_ε converges in $L^2(\mathbb{R}^n)$, to $u_0\chi_\Omega$. Since $\varphi_\varepsilon \equiv u_0$ in Ω, then by definition of the functionals (4.110), and (4.111), we can write

$$\mathcal{E}_{p,\varepsilon}(\varphi_\varepsilon) - \mathcal{E}_{p,0}(u_0) = \varepsilon \int_{\Sigma_\varepsilon} |\nabla\varphi_\varepsilon|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} \varphi_\varepsilon^2\, d\mathcal{H}^{n-1} - \beta \int_{\partial\Omega} \frac{u_0^2}{1+\beta h}\, d\mathcal{H}^{n-1}. \tag{4.115}$$

Computing the gradient of φ_ε, for any $x \in \Sigma_\varepsilon$,

$$|\nabla\varphi_\varepsilon|^2(x) \leq \frac{\beta^2 u_0^2(\sigma(x))}{\varepsilon^2(1+\beta h)^2} + C\big(|\nabla u_0(\sigma(x))|^2 + u_0^2(\sigma(x))\big),$$

where $C = C(h,\beta)$. Hence, by (3.43), and noticing that $\sigma + t\nu_0 \in \Gamma_{d_0}$ implies $d(\sigma + t\nu_0(\sigma)) = t$, we get

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} |\nabla\varphi_\varepsilon|^2\, dx \leq & \frac{\beta^2}{\varepsilon} \int_{\Sigma_\varepsilon} \frac{u_0^2(\sigma)}{(1+\beta h)^2}\, dx + \varepsilon C \int_{\Sigma_\varepsilon} \big(|\nabla u_0(\sigma(x))|^2 + u_0^2(\sigma(x))\big)\, dx \\ \leq & \beta^2 \int_{\partial\Omega} \frac{u_0^2(\sigma)h}{(1+\beta h)^2} \left(1 + \frac{\varepsilon h H}{2}\right) d\mathcal{H}^{n-1} + O(\varepsilon^2). \end{aligned} \tag{4.116}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$\varphi_\varepsilon(\sigma + \varepsilon h(\sigma)\nu_0(\sigma)) = \frac{u_0(\sigma)}{1+\beta h(\sigma)},$$

from which we get

$$\beta \int_{\partial\Omega_\varepsilon} \varphi_\varepsilon^2\, d\mathcal{H}^{n-1} \leq \beta \int_{\partial\Omega} \frac{u_0^2(\sigma)}{(1+\beta h)^2}(1 + \varepsilon h H)\, d\mathcal{H}^{n-1} + O(\varepsilon^2). \tag{4.117}$$

Finally, joining (4.115), (4.116), (4.117), we have

$$\begin{aligned}\delta\mathcal{E}_{p,\varepsilon}(\varphi_\varepsilon) = \frac{\mathcal{E}_{p,\varepsilon}(u_\varepsilon) - \mathcal{E}_{p,0}}{\varepsilon} &\leq \beta \int_{\partial\Omega} \frac{u_0^2(\sigma)hH}{(1+\beta h)^2}\left(\frac{\beta h}{2}+1\right) d\mathcal{H}^{n-1} + O(\varepsilon)\\ &= \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, u_0^2(\sigma)\, d\mathcal{H}^{n-1} + O(\varepsilon)\end{aligned}$$

so that

$$\limsup_{\varepsilon\to 0^+} \delta\mathcal{E}_{p,\varepsilon}(\varphi_\varepsilon) \leq \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, u_0^2(\sigma)\, d\mathcal{H}^{n-1}$$

and the Γ-limsup inequality is proved. □

As in subsection 4.2.2, in the following we will explicitly write the dependence on h of the solution to the limit Poisson problem (4.37), that is, we will write u_h in place of u_0. Let

$$\begin{aligned}\mathcal{G}_{p,\varepsilon}(h) &= \mathcal{E}_{p,0}(u_h, h) + \varepsilon\mathcal{E}_p^{(1)}(u_h, h)\\ &= -\int_\Omega f u_h\, dx + \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, u_h^2(\sigma)\, d\mathcal{H}^{n-1}.\end{aligned}$$

We are interested in the problem

$$\inf_{h\in\mathcal{H}_m} \mathcal{G}_{p,\varepsilon}(h). \tag{4.118}$$

In particular, we are interested in the existence of a solution and, in analogy to Theorem 3.24 (and Remark 3.25), how such a solution depends on the mean curvature H. Up until now, however, the problem remains open as the approximated energy $\mathcal{G}_{p,\varepsilon}$ does not appear to be lower semicontinuous with respect to the convergence in $\mathcal{H}_m$ (see Definition 4.30).

Chapter 5

The Spectral Problem

In this chapter, we will study the optimisation problem of finding the configuration of insulating material surrounding a conductor, which maintains the temperature the longest. As discussed in subsection 1.2.3, this type of problem can be related to the eigenvalue problem

$$\begin{cases} -\Delta u_A = \lambda_\beta(A) u_A & \text{in } \Omega, \\ -k\Delta u_A = \lambda_\beta(A) u_A & \text{in } A \setminus \overline{\Omega}, \\ \partial_{\nu_0} u_A^- = k \partial_{\nu_0} u_A^+ & \text{on } \partial\Omega, \\ k\partial_\nu u_A + \beta u_A = 0 & \text{on } \partial A, \end{cases}$$

where, as usual, Ω represent the conductor and A the insulated body. Then the best configuration of insulating material is the one that minimises the energy

$$\mathcal{E}_s(A) = \lambda_{\beta,1}(A) + C_0 \mathcal{L}^n\big(A \setminus \overline{\Omega}\big),$$

or, more generally,

$$\mathcal{E}_s(A) = \mathcal{F}(\lambda_{\beta,1}(A), \dots, \lambda_{\beta,j}(A)) + C_0 \mathcal{L}^n\big(A \setminus \overline{\Omega}\big),$$

for some $j \in \mathbb{N}$, and function $\mathcal{F} \colon \mathbb{R}^j \to \mathbb{R}$. In the one-phase setting, the optimisation of the Robin eigenvalues under volume constraint is well studied (see, for instance, [15, 33, 19, 20, 24] for the minimisation of the principal eigenvalue and [48, 21, 56] for the minimisation of higher eigenvalues). However, to our knowledge, no existence result is known in the two-phase setting. In the following, we will study the problem in the thin layer limit.

5.1 Asymptotic analysis

As in the previous chapters, let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded, open set, and let $h \colon \partial\Omega \to \mathbb{R}$ be a positive smooth function. Denoting by ν_0 the exterior unit normal to the boundary of Ω, for every $\varepsilon > 0$ sufficiently small, we define

$$\Sigma_\varepsilon = \{\, \sigma + t\nu_0(\sigma) \mid \sigma \in \partial\Omega,\ 0 < t < \varepsilon h(\sigma) \,\}$$

and we denote by $\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon$. Moreover, we will assume $k = \varepsilon$.

Let

$$0 < \lambda_{\varepsilon,1}(h) \le \lambda_{\varepsilon,2}(h) \le \dots \le \lambda_{\varepsilon,j}(h) \le \dots \to \infty$$

denote the spectrum of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ -\varepsilon \Delta u = \lambda u & \text{in } \Sigma_\varepsilon, \\ \partial_{\nu_0} u^- = \varepsilon \partial_{\nu_0} u^+ & \text{on } \partial\Omega, \\ \varepsilon \partial_\nu u + \beta u = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.1}$$

and let

$$0 < \lambda_1(h) \le \lambda_2(h) \le \cdots \le \lambda_j(h) \le \cdots \to \infty$$

denote the one of the eigenvalue problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_{\nu_0} u + \dfrac{\beta}{1+\beta h} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

In the following, we will discuss the convergence of $\lambda_{\varepsilon,j}(h)$ to $\lambda_j(h)$. Namely, in subsection 5.1.1 we will deal with the convergence of the eigenvalues, in subsection 5.1.2 we will discuss the optimisation of related energies, and, finally, in subsection 5.1.3, we will prove a first-order asymptotic estimate for the principal eigenvalue.

5.1.1 The limit eigenvalue problem

The convergence of the principal eigenvalue for this type of elliptic operators, in the case of the Dirichlet boundary condition, was studied in [42]. The case of the Robin boundary condition has been recently studied in [38]. In [30], we proved the convergence of the full spectrum using the abstract framework of [47].

Theorem 5.1 (Lemma 1 and Theorem 2 in [47])**.** *Let H_ε, H_0 be separable Hilbert spaces with scalar products $(\cdot,\cdot)_\varepsilon$, $(\cdot,\cdot)_0$ and norms $\|\cdot\|_\varepsilon$, $\|\cdot\|_0$ respectively. Consider $\mathcal{A}_\varepsilon : H_\varepsilon \to H_\varepsilon$ and $\mathcal{A}_0 : H_0 \to H_0$ continuous linear operators such that for all $w \in H_0$ $\mathcal{A}_0$, $v \in \mathcal{V}$, where $\mathcal{V}$ is a linear subspace of H_0. Assume the following:*

H1. There exist continuous linear operators $R_\varepsilon : H_0 \to H_\varepsilon$ such that for any $f \in \mathcal{V}$ we have

$$\lim_{\varepsilon\to 0^+} (R_\varepsilon f, R_\varepsilon f)_\varepsilon \to (f,f)_0,$$

H2. The operators $\mathcal{A}_\varepsilon$, $\mathcal{A}_0$ are positive, compact, and self-adjoint in $\mathcal{H}_\varepsilon$ and $\mathcal{H}_0$ respectively. Moreover

$$\sup_\varepsilon \|\mathcal{A}_\varepsilon\|_{\mathcal{L}(\mathcal{H}_\varepsilon)} < \infty.$$

H3. For any $f \in \mathcal{V}$ we have

$$\lim_{\varepsilon\to 0^+} \|\mathcal{A}_\varepsilon R_\varepsilon f - R_\varepsilon \mathcal{A}_0 f\|_\varepsilon = 0.$$

H4. The family of operators $\mathcal{A}_\varepsilon$ is uniformly compact in the following sense. Any sequence $f_\varepsilon \in \mathcal{H}_\varepsilon$ for which $\sup_\varepsilon \|f^\varepsilon\|_\varepsilon < \infty$ contains a subsequence f_{ε_k} such that for some $w_0 \in \mathcal{V}$,

$$\lim_{k\to+\infty} \|\mathcal{A}_{\varepsilon_k} f_{\varepsilon_k} - R_{\varepsilon_k} w_0\|_{\varepsilon_k} = 0.$$

Consider $\mu_{\varepsilon,1} \ge \mu_{\varepsilon,2} \ge \cdots \ge \mu_{\varepsilon,j} \ge \dots$ and $\mu_{0,1} \ge \mu_{0,2} \ge \dots \mu_{0,j} \ge \dots$ respectively the sequence of eigenvalues of $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$, where each eigenvalue is counted with multiplicity. Let $\{u_{\varepsilon,j}\}$ be an orthonormal basis of eigenvectors for $\mathcal{A}_\varepsilon$.

Then the following properties hold:

- *For every $j \in \mathbb{N}$,*
$$\lim_{\varepsilon\to 0^+} \mu_{\varepsilon,j} = \mu_{0,j}. \tag{5.3}$$

- *There exist ε_k a vanishing sequence and $\{u_j\}$ an orthonormal basis of eigenvector of $\mathcal{A}_0$ such that*
$$\lim_{k\to +\infty} \|u_{\varepsilon_k,j} - R_{\varepsilon_k} u_j\|_{\varepsilon_k} = 0. \tag{5.4}$$

- *Assume that μ is an eigenvalue of multiplicity l for $\mathcal{A}_0$ such that*
$$\mu = \mu_{0,j} = \mu_{0,j+1} = \cdots = \mu_{0,j+l}.$$
Then for all v such that $\mathcal{A}_0 v = \mu v$, there exists a family v_ε such that $v_\varepsilon \in span\{u_{\varepsilon,j}, \dots, u_{\varepsilon,j+l}\}$ and
$$\lim_{\varepsilon\to 0^+} \|v_\varepsilon - R_\varepsilon v\|_\varepsilon = 0. \tag{5.5}$$

Proposition 5.2. *Let $\{\lambda_{\varepsilon,j}\}$ and $\{\lambda_j\}$ denote the sequences of eigenvalues of problems* (5.1) *and* (5.2) *respectively, counted with multiplicity. Let $\{u_{\varepsilon,j}\}$ be an orthonormal basis of eigenfunctions associated with problem* (5.1)*. Then the following properties hold:*

- *For every $j \in \mathbb{N}$,*
$$\lim_{\varepsilon\to 0+} \lambda_{\varepsilon,j} = \lambda_j. \tag{5.6}$$

- *There exist ε_k a vanishing sequence and $\{u_j\}$ an orthonormal basis of eigenfunctions associated with problem* (5.2) *such that*
$$\lim_{k\to +\infty} \|u_{\varepsilon_k,j} - u_j\|_{L^2(\Omega)} = 0, \quad \lim_{k\to +\infty} \|u_{\varepsilon_k,j}\|_{L^2(\Sigma_{\varepsilon_k})} = 0. \tag{5.7}$$

- *Assume that λ is an eigenvalue of multiplicity l for problem* (5.2) *such that*
$$\lambda = \lambda_j = \lambda_{j+1} = \cdots = \lambda_{j+l}.$$
Then for all w such that w is an eigenfunction associated to λ for (5.2)*, there exists a family w_ε such that $w_\varepsilon \in span\{u_{\varepsilon,j}, \dots, u_{\varepsilon,j+l}\}$ and*
$$\lim_{\varepsilon\to 0^+} \|w_\varepsilon - w\|_{L^2(\Omega)} = 0, \quad \lim_{\varepsilon\to 0^+} \|w_\varepsilon\|_{L^2(\Sigma_\varepsilon)} = 0, \tag{5.8}$$

Proof. For every $\varepsilon > 0$ and $f \in L^2(\Omega_\varepsilon)$, let $u_{\varepsilon,f}$ be the solution to the Poisson problem
$$\begin{cases} -\Delta u_{\varepsilon,f} = f & \text{in } \Omega, \\ -\varepsilon\Delta u_{\varepsilon,f} = f & \text{in } \Sigma_\varepsilon, \\ \partial_{\nu_0} u^-_{\varepsilon,f} = \varepsilon \partial_{\nu_0} u^+_{\varepsilon,f} & \text{on } \partial\Omega, \\ \partial_\nu u_{\varepsilon,f} + \beta u_{\varepsilon,f} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.9}$$
and let
$$\mathcal{T}_\varepsilon \colon f \in L^2(\Omega_\varepsilon) \mapsto u_{\varepsilon,f} \in L^2(\Omega_\varepsilon)$$
be the resolvent operator. Analogously, for every $f \in L^2(\Omega)$ let u_f be the solution to the Poisson problem
$$\begin{cases} -\Delta u_f = f & \text{in } \Omega, \\ \partial_{\nu_0} u_f + \dfrac{\beta}{1+\beta h} u_f = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

and let

$$\mathcal{T}_0 \colon f \in L^2(\Omega) \mapsto u_f \in L^2(\Omega)$$

be the resolvent operator. It is well known that the operators $\mathcal{T}_\varepsilon$ and $\mathcal{T}_0$ are linear, continuous, compact, and self-adjoint. Moreover, their eigenvalues are given by the reciprocal of the eigenvalues $\lambda_{\varepsilon,j}$ and λ_j respectively. Then it is enough to prove that $\mathcal{T}_\varepsilon$ and $\mathcal{T}_0$ satisfy assumptions *H1*-*H4*.

To prove assumption *H1*, we just need to define the extension operators R_ε. In this case, we will see that, for every $f \in L^2(\Omega)$, it is enough to define

$$R_\varepsilon f = \begin{cases} f & \text{in } \Omega, \\ 0 & \text{in } \Sigma_\varepsilon. \end{cases}$$

Then assumption *H4* is the L^2 convergence of $u_{\varepsilon,f}$ to u_f when f is concentrated in Ω, which, as discussed in Remark 4.28, is a consequence of the Γ-convergence result of Theorem 4.26.

To prove assumption *H2*, recall the uniform Poincaré inequality (see Lemma 4.27)

$$\int_{\Omega_\varepsilon} v^2\,dx \le C\left[\int_\Omega |\nabla v|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla v|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1}\right],$$

for every $v \in H^1(\Omega_\varepsilon)$. Then

$$\begin{aligned}\int_{\Omega_\varepsilon} u_{\varepsilon,f}^2\,dx &\le C\left[\int_\Omega |\nabla u_{\varepsilon,f}|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla u_{\varepsilon,f}|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_{\varepsilon,f}^2\,d\mathcal{H}^{n-1}\right] \\ &= C\int_{\Omega_\varepsilon} f u_{\varepsilon,f}\,dx,\end{aligned}$$

where we used the fact that $u_{\varepsilon,f}$ is the solution to the Poisson problem (5.9). Then we can use Young's inequality and standard arguments to have

$$\int_{\Omega_\varepsilon} u_{\varepsilon,f}^2\,dx \le C^2\int_{\Omega_\varepsilon} f^2\,dx, \tag{5.11}$$

which is exactly *H2*. Notice that in addition, this also implies that,

$$\|u_{\varepsilon,f}\|_{H^1(\Omega)} \le C'\|f\|_{L^2(\Omega_\varepsilon)}. \tag{5.12}$$

Finally, to prove *H4*, let $f_\varepsilon \in L^2(\Omega_\varepsilon)$ such that

$$\sup_\varepsilon \|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} < +\infty.$$

Then, by (5.12), we have that $u_{\varepsilon,f_\varepsilon}$ are equibounded in $H^1(\Omega)$, so that, up to a subsequence, there exists a function $w \in H^1(\Omega)$, such that $u_{\varepsilon,f_\varepsilon}$ converges to w in $L^2(\Omega)$. We want to prove that $\|u_{\varepsilon,f_\varepsilon}\|_{L^2(\Sigma_\varepsilon)}$ goes to zero. To prove that, recall that, by the geometry of the problem (see Lemma 4.27), there exists a constant $C > 0$ such that, for every $v \in H^1(\Omega_\varepsilon)$

$$\int_{\Sigma_\varepsilon} v^2\,dx \le \varepsilon C\left[\varepsilon\int_{\Sigma_\varepsilon} |\nabla v|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1}\right].$$

Then applying the previous inequality to $u_{\varepsilon,f_\varepsilon}$, and using equation (5.9) and the estimate (5.11), we have

$$\int_{\Sigma_\varepsilon} u_{\varepsilon,f_\varepsilon}^2\,dx \le \varepsilon C'\int_{\Omega_\varepsilon} f_\varepsilon^2\,dx,$$

which proves the assumption. $\square$

We want to compare the previous result with some more known notions of convergence in perturbation theory (we refer to [62, section 9.3] for the definitions and to [34] for some applications to boundary value problems)

Definition 5.3. Let $\{\Omega_k\}, \Omega_0$ open subsets of $\mathbb{R}^n$ and identify the spaces $L^2(\Omega_k), L^2(\Omega_0)$ with the subspaces of $L^2(\mathbb{R}^n)$ of functions that are zero outside of Ω_k and Ω_0 respectively. Let P_k and P_0 are the functions $P_k f = f\chi_{\Omega_k}$ and $P_0 f = \chi_{\Omega_0}$ respectively. Let $\mathcal{B}_k$ and $\mathcal{B}_0$ linear (not necessary bounded), self-adjoint operators on $L^2(\Omega_k)$ and $L^2(\Omega_0)$. We will say that

- $\mathcal{B}_k$ converges to $\mathcal{B}_0$ in the *generalised strong resolvent* sense if, for some $z_0 \in \bigcup_k \rho(\mathcal{B}_k) \cap \rho(\mathcal{B}_0)$ in the intersection of the resolvent sets, and for every $f \in L^2(\mathbb{R}^n)$
$$(\mathcal{B}_k - z_0)^{-1} P_k f \to (\mathcal{B}_0 - z_0)^{-1} P_0 f,$$
in $L^2(\mathbb{R}^n)$,
- $\mathcal{B}_k$ converges to $\mathcal{B}_0$ in the *generalised norm resolvent* sense if, for some $z_0 \in \bigcup_k \rho(\mathcal{B}_k) \cap \rho(\mathcal{B}_0)$ in the intersection of the resolvent sets,
$$(\mathcal{B}_k - z_0)^{-1} P_k \to (\mathcal{B}_0 - z_0)^{-1} P_0,$$
in the operator norm.

While, in general, the strong resolvent convergence does not imply the convergence of the eigenvalues to the corresponding eigenvalues of the limit problem, properties like (5.6) are true for norm resolvent convergence. If we denote by $\mathcal{B}_\varepsilon$ and $\mathcal{B}_0$ the elliptic operators associated with the boundary value problems (5.9) and (5.10), respectively, it is clear that $\mathcal{B}_\varepsilon$ converges to $\mathcal{B}_0$ in the generalised strong resolvent sense, while it does not appear to converge in the generalised norm resolvent one, hence the convergence in Proposition 5.2 is stronger than the generalised strong resolvent convergence and weaker then the generalise norm resolvent one.

5.1.2 Optimisation of spectral energies

We now turn our attention to the Optimisation problems of the type
$$\min_{h \in \mathcal{H}_m} \mathcal{F}(\lambda_1(h), \dots, \lambda_j(h)), \tag{5.13}$$
where we recall that
$$\mathcal{H}_m = \left\{ h \in L^1(\partial\Omega) \;\middle|\; \begin{array}{l} \displaystyle\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} \leq m \\ h \geq 0 \end{array} \right\}.$$
We start with the optimisation of the principal eigenvalue $\lambda_1(h)$ in [38].

Theorem 5.4. *For every $m > 0$, the optimisation problem*
$$\lambda_m = \min_{h \in \mathcal{H}_m} \lambda_1(h) \tag{5.14}$$
admits a solution h_m. Moreover, h_m saturates the integral constraint, that is
$$\int_{\partial\Omega} h_m \, d\mathcal{H}^{n-1} = m,$$
and can be written as
$$h_m(\sigma) = \begin{cases} \dfrac{|u_m(\sigma)|}{c_m \beta} - \dfrac{1}{\beta} & \text{if } |u_m(\sigma)| \geq c_m, \\ 0 & \text{otherwise,} \end{cases}$$

where c_m is the unique constant which satisfies

$$c_m = \frac{1}{\mathcal{H}^{n-1}(\partial\Omega \cap \{\, |u_m| \geq c_m \,\}) + m\beta} \int_{\partial\Omega\cap\{\,|u_m|\geq c_m\,\}} |u_m| \, d\mathcal{H}^{n-1},$$

and u_m is an eigenfunction of the problem

$$\begin{cases} -\Delta u_m = \lambda_m u_m & \text{in } \Omega, \\ \partial_{\nu_0} u_m + \dfrac{\beta}{1+\beta h_m} u = 0 & \text{on } \partial\Omega. \end{cases}$$

The proof of the previous theorem is based on the optimisation of the Rayleigh quotient

$$R(v,h) = \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \beta \int_{\partial\Omega} \frac{v^2}{1+\beta h} \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2 \, dx},$$

among $(v,h) \in H^1(\Omega) \times \mathcal{H}_m$, and follows the approach of [36] for the proof of Theorem 4.29. Moreover, the authors prove that the function

$$m \in (0,+\infty) \mapsto \lambda_m \in (0,+\infty)$$

is continuous, decreasing and that

$$\lim_{m\to\infty} \lambda_m = 0, \quad \text{and} \quad \lim_{m\to 0^+} \lambda_m^D = \lambda_{\beta,1}(\Omega),$$

where $\lambda_{\beta,1}(\Omega)$ is the first eigenvalue of the Robin Laplacian on Ω.

To study problem (5.13), in [30], we prove the following continuity result with respect to the convergence in $\mathcal{H}_m$ (see Definition 4.30).

Theorem 5.5. *Let $\{h_\varepsilon\} \subset \mathcal{H}_m$ such that h_ε converges to h_0 in $\mathcal{H}_m$. Let $\{\lambda_j(h_\varepsilon)\}$ and $\{\lambda_j(h_0)\}$ denote the sequences of eigenvalues of problems* (5.2)*, counted with multiplicity, and let $\{u_{h_\varepsilon,j}\}$ be an orthonormal basis of eigenfunctions associated with the eigenvalues $\{\,\lambda_j(h_\varepsilon)\,\}$. Then the following properties hold:*

- *For every $j \in \mathbb{N}$,*
$$\lim_{\varepsilon\to 0^+} \lambda_j(h_\varepsilon) = \lambda_j(h_0).$$
- *There exist ε_k a vanishing sequence and $\{u_{h_0,j}\}$ an orthonormal basis of eigenfunctions associated with problem* (5.2) *such that*
$$\lim_{k\to+\infty} \|u_{h_{\varepsilon_k},j} - u_{h_0,j}\|_{L^2(\Omega)} = 0. \tag{5.15}$$
- *Assume that λ is an eigenvalue of multiplicity l for problem* (5.2) *such that*
$$\lambda = \lambda_j(h_0) = \lambda_{j+1}(h_0) = \cdots = \lambda_{j+l}(h_0).$$
Then for all w such that w is an eigenfunction associated to λ for (5.2)*, there exists a family w_ε such that $w_\varepsilon \in span\{u_{h_\varepsilon,j}, \ldots, u_{h_\varepsilon,j+l}\}$ and*
$$\lim_{\varepsilon\to 0^+} \|w_\varepsilon - w\|_{L^2(\Omega)} = 0. \tag{5.16}$$

Proof. The assertion follows from Theorem 5.1. In particular, for every $h \in \mathcal{H}_m$ and for every $f \in L^2(\Omega)$ let $u_{h,f}$ be the solution to the Poisson problem

$$\begin{cases} -\Delta u_f = f & \text{in } \Omega, \\ \partial_{\nu_0} u_f + \dfrac{\beta}{1+\beta h} u_f = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.17}$$

and let

$$\mathcal{T}_h \colon f \in L^2(\Omega) \mapsto u_{h,f} \in L^2(\Omega)$$

be the associated resolvent operator. Then we have to prove that, if h_ε converges to h_0 in $\mathcal{H}_m$,then $\mathcal{T}_{h_\varepsilon}$ and $\mathcal{T}_{h_0}$ satisfy assumptions *H1 - H4*.

Assumption *H1* trivially holds true, as the Hilbert space does not vary, so that R_ε is the identity. Assumptions *H1-H4* follow from the continuity result of $u_{h,f}$ in h (Proposition 4.32) and the uniform Poincaré inequality (4.47) with a similar proof as Proposition 5.2. □

We then have the following proposition.

Proposition 5.6. *Let $j \in \mathbb{N}$, let $\mathcal{F} \colon \mathbb{R}^j \to \mathbb{R}$ be a lower semicontinous function and Fix $i_1, \dots, i_j \in \mathbb{N}$. Then the problem*

$$\min_{h \in \mathcal{H}_m} \mathcal{F}(\lambda_{i_1}(h), \dots, \lambda_{i_j}(h))$$

admits a solution h_0. Moreover, if $\mathcal{F}$ is increasing in every variable, h_0 saturates the integral constraint, that is

$$\int_{\partial\Omega} h_0 \, d\mathcal{H}^{n-1} = m.$$

Proof. The existence result is a direct consequence of the lower semicontinuity of $\mathcal{F}$, the continuity result of Theorem 5.5 and the compactness of $\mathcal{H}_m$.

To prove that, if $\mathcal{F}$ is increasing in each variable, then h_0 saturates the constraint, it is enough to show that each eigenvalue λ_j is decreasing in h. Indeed, let h_1 and h_2 in $\mathcal{H}_m$ such that $h_1 \le h_2$, then, in the notation of Definition 4.30, $b_{h_1} \ge b_{h_2}$. Using the min-max characterization of the eigenvalues, there exists $\bar{V}$ a $j-$dimensional subspace of $H^1(\Omega)$ such that

$$\begin{aligned} \lambda_j(h_1) &= \max_{v \in \bar{V} \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} b_{h_1} v^2 \, d\mathcal{H}^{n-1}}{\int_\Omega v^2 \, dx} \\ &\ge \max_{v \in \bar{V} \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} b_{h_2} v^2 \, d\mathcal{H}^{n-1}}{\int_\Omega v^2 \, dx} \\ &\ge \min_{V \in G_j} \max_{v \in V \setminus \{0\}} \frac{\int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} b_{h_2} v^2 \, d\mathcal{H}^{n-1}}{\int_\Omega v^2 \, dx} = \lambda^j(h_2), \end{aligned}$$

which proves the monotonicity. □

5.1.3 First-order development

We now prove a first-order asymptotic development, in ε of the principal eigenvalue $\lambda_{\varepsilon,1}(h)$. Namely, let H be the mean curvature of the boundary of Ω, for every $v \in L^2(\partial\Omega)$ and $\lambda \in \mathbb{R}$, let

$$R^{(1)}(v, \lambda) = \int_{\partial\Omega} \frac{\beta H h(2+\beta h)}{2(1+\beta h)^2} v^2 \, d\mathcal{H}^{n-1} - \lambda \int_{\partial\Omega} \frac{h(3+3\beta h+\beta^2 h^2)}{3(1+\beta h)^2} v^2 \, d\mathcal{H}^{n-1},$$

we have the following theorem.

Theorem 5.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $h(x) = h(\sigma(x))$. Then*

$$\lim_{\varepsilon \to 0^+} \frac{\lambda_{\varepsilon,1}(h) - \lambda_1(h)}{\varepsilon} = R^{(1)}(u_1, \lambda_1(h)), \tag{5.18}$$

where u_1 is the eigenfunction associated to $\lambda_1(h)$ with $\|u_1\|_{L^2(\Omega)} = 1$.

The idea of the theorem is similar to the one of Theorem 4.36 and is based on a simple generalisation of Theorem 4.35. Let $\Psi_\varepsilon \in C^{0,1}(\Gamma_{d_0}; \Gamma_{\varepsilon d_0})$ be the stretching diffeomorphism defined in Definition 4.37 as

$$\Psi_\varepsilon(z) = \sigma(z) + \varepsilon d(z) \nu_0(z).$$

For every $f \in L^2(\mathbb{R}^n)$ let $u_{\varepsilon,f}$ and u_f be the solutions to the Poisson problems

$$\begin{cases} -\Delta u_{\varepsilon,f} = f & \text{in } \Omega, \\ -\varepsilon \Delta u_{\varepsilon,f} = f & \text{in } \Sigma_\varepsilon, \\ \partial_{\nu_0} u^-_{\varepsilon,f} = \varepsilon \partial_{\nu_0} u^+_{\varepsilon,f} & \text{on } \partial\Omega, \\ \partial_\nu u_{\varepsilon,f} + \beta u_{\varepsilon,f} = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta u_f = f & \text{in } \Omega, \\ \partial_{\nu_0} u_f + \dfrac{\beta}{1+\beta h} u_f = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively. We have the following theorem.

Theorem 5.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $h(x) = h(\sigma(x))$. Let $f_\varepsilon \in L^2(\Omega_\varepsilon)$ and assume that there exist $C_f > 0$ such that for every $\varepsilon > 0$*

$$\int_{\Omega_\varepsilon} f_\varepsilon^2 \, dx \le C_f, \tag{5.19}$$

and

$$\int_{\Sigma_\varepsilon} f_\varepsilon^2 \, dx \le \varepsilon C_f. \tag{5.20}$$

Let $u_\varepsilon = u_{\varepsilon, f_\varepsilon}$ and

$$\tilde{u}_\varepsilon(z) = \begin{cases} u_\varepsilon(z) & \text{if } z \in \Omega, \\ u_\varepsilon(\Psi_\varepsilon(z)) & \text{if } z \in \Sigma_1. \end{cases}$$

Then the family $\{\, \tilde{u}_\varepsilon \,\}$ is equibounded in $H^1(\Omega_1)$. Moreover, up to a subsequence, f_ε converges weakly in L^2 to a function f with $f = 0$ almost everywhere in $\mathbb{R}^n \setminus \Omega$ and, up to an other subsequence, $\tilde{u}_\varepsilon$ converges weakly in $H^1(\Omega_1)$, as ε goes to 0, to the function

$$\tilde{u}_f(z) = \begin{cases} u_f(z) & \text{if } z \in \Omega, \\ u_f(\sigma(z)) \left(1 - \dfrac{\beta d(z)}{1 + \beta h(z)} \right) & \text{if } z \in \Sigma_1. \end{cases} \tag{5.21}$$

We remark that if f_ε is not convergent, then the limit might depend on the subsequence.

Proof. The main difference with Theorem 4.35 is that now the right-hand side, f_ε, does not vanish in Σ_ε. However, the non-concentration condition (5.20) and the equiboundedness condition (5.19) are sufficient to prove the energy estimates Theorem 4.42 and to compute the limit equation as in Proposition 4.39. For simplicity's sake, we omit the details of the proof. □

Choosing $f_\varepsilon = \lambda_{\varepsilon,j} u_{\varepsilon,j}$, we can apply the previous result to eigenfunctions. Namely, we have the following corollary.

Corollary 5.9. *Let $u_{\varepsilon,j}$ be a sequence of normalised eigenfunctions of eigenvalue $\lambda_{\varepsilon,j}(h)$. Then, up to a subsequence, the functions $\tilde{u}_{\varepsilon,j} = u_{\varepsilon,j} \circ \Psi_\varepsilon$ converge, weakly in $H^1(\Omega_1)$ and strongly in $H^1(\Omega)$ to*

$$\tilde{v}_j(z) = \begin{cases} v_j(z) & \text{if } z \in \Omega, \\ v_j(\sigma(z))\left(1 - \dfrac{\beta d(z)}{1+\beta h(z)}\right) & \text{if } z \in \Sigma_1, \end{cases}$$

where v_j is a normalised eigenfunction of eigenvalue $\lambda_j(h)$.

We can finally prove Theorem 5.7.

proof of Theorem 5.7. We prove the theorem by evaluating the liminf and limsup separately.

Let $u_\varepsilon = u_{\varepsilon,1}$ be the sequence of non-negative, normalised, first eigenfunctions. By the variational characterisation

$$\lambda_1(h) \leq \frac{\displaystyle\int_\Omega |\nabla u_\varepsilon|^2\,dx + \int_{\partial\Omega} \frac{\beta}{1+\beta h} u_\varepsilon^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega u_\varepsilon^2},$$

while

$$\lambda_{\varepsilon,1}(h) = \int_\Omega |\nabla u_\varepsilon|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1}.$$

Then

$$\begin{aligned}\frac{\lambda_{\varepsilon,1}(h) - \lambda_1(h)}{\varepsilon} \geq& \frac{1}{\varepsilon}\left[\varepsilon\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\,d\mathcal{H}^{n-1} - \int_{\partial\Omega}\frac{\beta}{1+\beta h}u_\varepsilon^2\,d\mathcal{H}^{n-1}\right] \\ &- \left[\int_\Omega |\nabla u_\varepsilon|^2\,dx + \int_{\partial\Omega}\frac{\beta u_\varepsilon^2}{1+\beta h}\,d\mathcal{H}^{n-1}\right]\frac{1}{\varepsilon}\int_{\Sigma_\varepsilon} u_\varepsilon^2\,dx.\end{aligned}$$

Let ε_k such that

$$\lim_{k\to+\infty} \frac{\lambda_{\varepsilon_k,1}(h) - \lambda_1(h)}{\varepsilon_k} = \liminf_{\varepsilon\to 0^+} \frac{\lambda_{\varepsilon,1}(h) - \lambda_1(h)}{\varepsilon}.$$

By Corollary 5.9 we have that, up to a further subsequence, u_{ε_k} converges in $H^1(\Omega)$ to the non-negative normalised eigenfunction of $\lambda_1(h)$, u_1 so that

$$\lim_{k\to+\infty} \int_\Omega |\nabla u_{\varepsilon_k}|^2\,dx + \int_{\partial\Omega} \frac{\beta u_{\varepsilon_k}^2}{1+\beta h}\,d\mathcal{H}^{n-1} = \lambda_1(h). \tag{5.22}$$

From Lemma 4.38 and Remark 3.16, we have

$$\frac{1}{\varepsilon_k}\int_{\Sigma_{\varepsilon_k}} u_{\varepsilon_k}^2\,dx = \int_{\partial\Omega}\int_0^{h(\sigma)} \tilde{u}_{\varepsilon_k}^2(\sigma + s\nu) J(\varepsilon_k s, \sigma)\,ds\,d\mathcal{H}^{n-1}$$

where $J(\varepsilon s, \sigma) \to 1$ uniformly for $\varepsilon \to 0^+$. Then, using again Corollary 5.9, (up to a subsequence) $\tilde{u}_{\varepsilon_k}$ converge to $\tilde{u}_1$ in $L^2(\Sigma_1)$, so that

$$\lim_{k\to+\infty} \frac{1}{\varepsilon_k}\int_{\Sigma_{\varepsilon_k}} u_{\varepsilon_k}^2 = \int_{\partial\Omega}\int_0^{h(\sigma)} \tilde{u}_1^2(\sigma + s\nu)\,ds\,d\mathcal{H}^{n-1},$$

which can be explicitly calculated as

$$\int_0^{h(\sigma)} \tilde{u}^2(\sigma + s\nu)\,ds = u(\sigma)^2 \int_0^{h(\sigma)} \left(1 - \frac{\beta s}{1+\beta h(\sigma)}\right)^2 ds = \frac{h(\sigma)u(\sigma)^2(3 + 3\beta h + \beta^2 h^2)}{3(1+\beta h)^2}.$$

So that

$$\lim_{k\to+\infty}\frac{1}{\varepsilon_k}\int_{\Sigma_{\varepsilon_k}} u_{\varepsilon_k}^2\,dx=\int_{\partial\Omega}\frac{h(3+3\beta h+\beta^2h^2)}{3(1+\beta h)^2}u^2\,d\mathcal{H}^{n-1}. \tag{5.23}$$

On the other hand, let

$$A_\varepsilon=\frac{1}{\varepsilon}\left[\varepsilon\int_{\Sigma_\varepsilon}|\nabla u_\varepsilon|^2\,dx+\beta\int_{\partial\Omega_\varepsilon}u_\varepsilon^2\,d\mathcal{H}^{n-1}-\int_{\partial\Omega}\frac{\beta}{1+\beta h}u_\varepsilon^2\,d\mathcal{H}^{n-1}\right],$$

arguing as in Theorem 4.36, and using Corollary 5.9 we can prove that

$$\lim_{k\to+\infty}A_{\varepsilon_k}\geq\int_{\partial\Omega}\frac{\beta Hh(2+\beta h)}{2(1+\beta h)^2}u_1^2\,d\mathcal{H}^{n-1}. \tag{5.24}$$

Finally, joining (5.24), (5.22) and (5.23), we have

$$\liminf_{\varepsilon\to0^+}\frac{\lambda_{\varepsilon,1}(h)-\lambda_1(h)}{\varepsilon}\geq R^{(1)}(u_1,\lambda_1(h)).$$

Let now

$$\varphi_\varepsilon(x)=\begin{cases}u(x) & \text{if } x\in\Omega,\\ u(\sigma(x))\left(1-\dfrac{\beta d(x)}{\varepsilon(1+\beta h(x))}\right) & \text{if } x\in\Sigma_\varepsilon,\\ 0 & \text{if } x\in\mathbb{R}^n\setminus\Omega_\varepsilon,\end{cases}$$

Then, using φ_ε as a test function in the variational characterisation of $\lambda_{\varepsilon,1}(h)$, and using the fact that $\varphi_\varepsilon=u_1$ in Ω, we have

$$\frac{\lambda_\varepsilon(h)-\lambda(h)}{\varepsilon}\leq\frac{\varepsilon\displaystyle\int_{\Sigma_\varepsilon}|\nabla\varphi_\varepsilon|^2\,dx+\beta\int_{\partial\Omega_\varepsilon}\varphi_\varepsilon^2d\mathcal{H}^{n-1}-\int_{\partial\Omega}\frac{\beta}{1+\beta h}u^2\,d\mathcal{H}^{n-1}}{\varepsilon\left(1+\displaystyle\int_{\Sigma_\varepsilon}\varphi_\varepsilon^2\,dx\right)}$$

$$-\lambda_1(h)\frac{\displaystyle\int_{\Sigma_\varepsilon}\varphi_\varepsilon^2\,dx}{\varepsilon\left(1+\displaystyle\int_{\Sigma_\varepsilon}\varphi_\varepsilon^2\,dx\right)}.$$

Then arguing as in Theorem 4.36, using the fact that φ_ε converges to $u_1\chi_\Omega$ in L^2, and by direct computation, we finally have

$$\limsup_{\varepsilon\to0^+}\frac{\lambda_{\varepsilon,1}(h)-\lambda_1(h)}{\varepsilon}\leq R^{(1)}(u_1,\lambda_1(h)).$$

□

Remark 5.10. We notice that if Ω is not connected, then the eigenvalue $\lambda_1(h)$ might not be simple. In this case, the theorem still holds in the following form:

$$\lim_{\varepsilon}\frac{\lambda_{\varepsilon,1}(h)-\lambda_1(h)}{\varepsilon}=\min_{v\in E_1(h)}R^{(1)}(v,\lambda_1(h)),$$

where $E_1(h)$ is the set of all L^2 normalised eigenfunctions associated with $\lambda_1(h)$. Using the min-max characterisation of the eigenvalues, we can prove that the theorem still holds for the generic j-th eigenvalue in the following weaker form (see [30]):

$$\liminf_{\varepsilon} \frac{\lambda_{\varepsilon,j}(h) - \lambda_j(h)}{\varepsilon} \geq \min_{v \in E_j(h)} R^{(1)}(v, \lambda_j(h)),$$

and

$$\limsup_{\varepsilon} \frac{\lambda_{\varepsilon,j}(h) - \lambda_j(h)}{\varepsilon} \leq \max_{v \in E_j(h)} R^{(1)}(v, \lambda_j(h)),$$

where $E_j(h)$ is the set of all L^2 normalised eigenfunctions associated with $\lambda_j(h)$.

5.2 The symmetry breaking phenomenon

The most interesting and surprising result on this topic is probably the *symmetry breaking* result by [18] for the Dirichlet boundary condition (see also [13, 46, 45]), which has been recently generalised to the Robin one in [38]. As the two results are closely related, we will start by briefly introducing the Dirichlet case.

Let $\lambda^D_{\varepsilon,1}(h)$ be the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ -\varepsilon \Delta u = \lambda u & \text{in } \Sigma_\varepsilon, \\ \partial_{\nu_0} u^- = \varepsilon \partial_{\nu_0} u^+ & \text{on } \partial\Omega, \\ u = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{5.25}$$

By the results of [42],

$$\lim_{\varepsilon \to 0^+} \lambda^D_{\varepsilon,1}(h) = \lambda^D_1(h),$$

which is the first eigenvalue of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \partial_{\nu_0} u + \dfrac{1}{h} u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.26}$$

Notice that problem (5.26) is exactly the limit problem, for β going to infinity, of the eigenvalue problem (5.2).

$\lambda^D_1(h)$ can be variationally characterised as

$$\lambda^D_1(h) = \min_{v \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} \frac{v^2}{h} \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2}.$$

Then, in [18], the authors study the optimisation of the Rayleigh quotient

$$R^D(v, h) = \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \int_{\partial\Omega} \frac{v^2}{h} \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2}$$

among $(v, h) \in H^1(\Omega) \times \mathcal{H}_m$, and prove the following theorem.

Theorem 5.11. *For every $m > 0$, the problem*

$$\lambda_m^D = \min_{h \in \mathcal{H}_m} \lambda_1^D(h), \tag{5.27}$$

admits a solution h_m^D which can be written as

$$h_m^D = m \frac{|u_m^D|}{\displaystyle\int_{\partial\Omega} |u_m^D| \, d\mathcal{H}^{n-1}},$$

where u_m^D is an eigenfunction of the problem

$$\begin{cases} -\Delta u_m^D = \lambda_m^D u_m^D & \text{in } \Omega, \\ \partial_{\nu_0} u_m^D + \dfrac{1}{h_m} u_m^D = 0 & \text{on } \partial\Omega. \end{cases}$$

In analogy with Theorem 4.34, we remark that the function u_m in the previous theorem is also a solution to the auxiliary problem

$$\lambda_m^D = \min_{v \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx + \frac{1}{m}\left(\int_{\partial\Omega} |v| \, d\mathcal{H}^{n-1}\right)^2}{\displaystyle\int_\Omega v^2}. \tag{5.28}$$

Moreover, the function

$$m \in (0, +\infty) \mapsto \lambda_m^D \in (0, +\infty)$$

is continuous, decreasing, and we have

$$\lim_{m \to \infty} \lambda_m^D = 0, \quad \text{and} \quad \lim_{m \to 0^+} \lambda_m^D = \lambda_1^D(\Omega),$$

where $\lambda_1^D(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian on Ω.

We can now state the symmetry breaking result.

Theorem 5.12. *Let Ω be a ball. There exists $m_0^D > 0$ such that the solution, u_m^D, to the auxiliary problem* (5.28) *is radial if $m > m_0^D$ and it is not radial if $0 < m < m_0^D$. That is, the solution h_m^D to problem* (5.27) *is not constant when $0 < m < m_0^D$.*

The value m_0^D is the unique solution to

$$\lambda_m^D = \lambda_2^N(\Omega),$$

where $\lambda_2^N(\Omega)$ is the first non-zero eigenvalue of the Neumann Laplacian on Ω. In [46], the value was explicitly computed as

$$m_0^D = \frac{n-1}{n\lambda_2^N(\Omega)} \frac{P(\Omega)^2}{\mathcal{L}^n(\Omega)}.$$

Notice that, by the Theorem 5.12, when Ω is a ball, for $m > m_0^D$, the optimiser h_m^D is constant so that the value λ_m^D is exactly the Robin eigenvalue with boundary parameter

$$\beta_m = \frac{1}{h_m^D} = \frac{P(\Omega)}{m}.$$

Then, by the continuity of λ_m^D and $\lambda_{\beta,1}(\Omega)$, we have that, if Ω is a ball, then the unique solution to

$$\lambda_{\beta,1}(\Omega) = \lambda_2^N(\Omega)$$

is given by

$$\beta^* = \frac{n\mathcal{L}^n(\Omega)}{(n-1)P(\Omega)}\lambda_2^N(\Omega) = \frac{\lambda_2^N(\Omega)}{H_\Omega}$$

The symmetry breaking result in [38] is a generalisation of Theorem 5.12

Theorem 5.13. *Let Ω be a ball. For every $\beta > \beta^*$ there exists $m_\beta > 0$ such that the h_m is the solution to problem* (5.14) *and u_m an associated eigenfunction as in Theorem 5.4, then*

- *if $0 < \beta < \beta^*$, u_m is radial and h_m is constant;*
- *if $\beta > \beta^*$ and $0 < m < m_\beta$, u_m is not radial and h_m is not constant;*
- *if $\beta > \beta^*$ and $m > m_\beta$, u_m is radial and h_m is constant.*

The critical value m_β is once again defined as the solution to

$$\lambda_m = \lambda_2^N(\Omega),$$

however, in this case $\lambda_m < \lambda_{\beta,1}(\Omega)$, so that the previous equation has a solution only when

$$\lambda_{\beta,1}(\Omega) > \lambda_2^N(\Omega),$$

that is when $\beta \geq \beta^*$. As before, by continuity, we have that

$$m_\beta = P(\Omega)\left(\frac{1}{\beta^*} - \frac{1}{\beta}\right) = m_0^D - \frac{P(\Omega)}{\beta}.$$

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